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Finite Element Methods of Dirichlet Boundary Optimal Control Problems With Weakly Imposed Boundary Conditions

Cagnur Corekli

Ph.D.

University of Connecticut, 2016

ABSTRACT

In the present work, we consider Symmetric Interior Penalty Galerkin (SIPG) method to approximate the solution to Dirichlet optimal control problem governed by a linear advection-diffusion-reaction equation on a convex polygonal domain.

The main feature of the method is that Dirichlet boundary conditions enter naturally into bilinear form and the finite element analysis can be performed in the standard setting. Another advantage of the method is that the method is stable and can be of arbitrary high degree. We show existence and uniqueness of the analytical and discrete solutions of the problem and derive optimal error estimates for the control on general convex polygonal domains.

Finally, we support our main results and highlight some of the features of the method with the several numerical examples in one and two dimensions. We also investigate numerically the performance of the method for advection-dominated problems.

Finite Element Methods of Dirichlet Boundary Optimal Control Problems With Weakly Imposed Boundary Conditions

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2016

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2016

APPROVAL PAGE

Doctor of Philosophy Dissertation

**Finite Element Methods of Dirichlet Boundary
Optimal Control Problems With Weakly
Imposed Boundary Conditions**

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2016

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stated to describe his presence for me while I can not find words to describe how grateful I am to have him as an advisor. All I can say is, he was an excellent mentor and advisor that I could have chosen to do Ph.D. I am indebted to him for all his priceless guidance and support and I always deeply appreciate his precious help. As a famous saying of Rumi, *I know you are tired but come, this is the way*. Definitely, the way is life and we are tired voyagers, but the most meaningful thing in this life is to choose the right people to walk together and he is the rightest person I can choose on my way.

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DEDICATION

It is an inexpressible pleasure for me to dedicate this dissertation to my mother Unzule Corekli and my father Hidayet Corekli to just give them as a present for their unconditional love and numerous sacrifices they have done for me since I was born.

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Chapter 1

Introduction

1.1 What is optimal control?

Optimal control theory is multidisciplinary field that requires knowledge from several areas of mathematics such as numerical and continuous optimization, theory of PDE's, numerical analysis, linear and nonlinear functional analysis and etc. The aim of the optimal control problem with the PDE's constraints is to minimize the cost functional of a controlled system described by partial differential equations. We illustrate the idea on examples from the textbook of Tröltzsch [32]. A vehicle that a time $t = 0$ is at the space point A moves along a straight line and stops at time $T > 0$ at another point B on that line. Suppose that the vehicle can be accelerated along the line in both forward and backward directions. What is the minimal time $T > 0$ needed for the travel, provided that the available thrust $q(t)$ at time t is subject to the constraints maximum backward and forward accelerations $-1 \leq q(t) \leq 1$?

Here is the model of the problem;

$y(t)$ denotes the position of the vehicle at time t , m is the mass of vehicles, $y_0, y_T \in \mathbb{R}$ are corresponding to positions of the initial points A and B. Then minimize $T > 0$ subject to the constraints for $\forall t \in [0, T]$

$$my''(t) = q(t),$$

$$y(0) = y_0,$$

$$y'(0) = 0,$$

$$y(T) = y_T,$$

$$y'(T) = 0,$$

$$|q(t)| \leq 1.$$

The features of the optimal control problem are come out as the following

1. a *cost functional* J to be minimized,
2. an initial values of the problem state the motion to determine the state y ,
3. a *control function* q ,
4. *various constraints*.

In general, the input (*control*) can be a function, boundary conditions, initial conditions, a coefficient of a system of equations, or a parameter in the equation. The output is usually the solutions of differential equations or system of equations. The control is chosen free within the given constraints and has to be chosen in desired way that the cost function is minimized, such controls are called *optimal*, and the output is called the *state* of the system. The state is some linear or nonlinear operator of the control with the assumption that there is a unique state for each control. Thus, the aim of the problem is to minimize a cost functional depending on the observation of the state and on the control. Here, we have some

examples of the optimal control problem with PDE's.

1.1.1 Some examples of the optimal control problem

- **Optimal stationary problem**

Consider a body Ω in \mathbb{R}^3 that is to be heated or cooled. We apply to its boundary Γ a heat source $q(x)$ (the control) that depends on the location x on the boundary. Our aim is to choose the control in such a way that corresponding temperature distribution $y(x)$ in the domain is the best possible approximation to the desired stationary temperature distribution $\hat{y}(x)$. The model of the problem is as the following:

$$\min J(y, q) := \frac{1}{2} \|y(x) - \hat{y}(x)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|q(x)\|_{L^2(\Gamma)}^2,$$

subject to

$$-\Delta y = 0 \quad \text{in } \Omega,$$

$$\frac{\partial y}{\partial \nu} = q \quad \text{on } \Gamma,$$

with control constraints

$$q_a(x) \leq q(x) \leq q_b(x) \quad \text{on } \Gamma.$$

This problem can be classified as *linear- quadratic elliptic boundary control problem*.

Let us give another example of the distributed control problem.

- **Optimal heat source problem**

The body Ω is heated by electromagnetic induction or by microwaves, so the control act as *heat source* in Ω , and temperature distribution in the domain is $y(x)$. Our aim is again to choose the control in such a way that corresponding temperature

distribution $y(x)$ in the domain is the best possible approximation to the desired stationary temperature distribution $\hat{y}(x)$. By assuming that the boundary temperature vanishes, we obtain the following problem

$$\min J(y, q) := \frac{1}{2} \|y(x) - \hat{y}(x)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|q(x)\|_{L^2(\Omega)}^2$$

subject to

$$-\Delta y = \beta q \quad \text{in } \Omega,$$

$$y = 0 \quad \text{on } \Gamma$$

and

$$q_a(x) \leq q(x) \leq q_b(x) \quad \text{in } \Omega.$$

where $\beta = \chi_{\Omega_c}$ and $\Omega_c \subset \Omega$. Here, q acts only in a subdomain $\Omega_c \subset \Omega$ because of the special choice β , and this problem is classified as *linear-quadratic elliptic control problem with distributed control*.

1.2 Preface

In this thesis, we consider the Dirichlet boundary optimal control problem. Let Ω be a convex polygonal domain in \mathbb{R}^2 and assume that the model of the optimal control problem has the following structure;

$$\min_{\{y, q\}} J(y, q) = \frac{1}{2} \|y - \hat{y}\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|q\|_{L^2(\Gamma)}^2 \quad (1.2.1)$$

subject to the advection-diffusion equation

$$-\Delta y(x) + \vec{\beta}(x) \cdot \nabla y(x) + c(x)y(x) = f(x), \quad x \in \Omega, \quad (1.2.2a)$$

$$y(x) = q(x), \quad x \in \Gamma. \quad (1.2.2b)$$

Here, $y(x)$ denotes the state variable, (1.2.2a) and (1.2.2b) is called the state equation, $q(x)$ is the control, $\Gamma = \partial\Omega$.

We assume that $f(x), \hat{y}(x) \in L^2(\Omega)$, $\vec{\beta}(x) \in [W_\infty^1(\Omega)]^2$, $c(x) \in L^\infty(\Omega)$ with the assumption $c(x) - \frac{1}{2}\nabla \cdot \vec{\beta}(x) \geq 0$, and $\alpha > 0$ is a given scalar.

This problem is important in many applications, for example distribution of pollution in air [35] or water [1] and for problems in computational electro-dynamics, gas and fluid dynamics [6]. However, there are several challenges involved in solving this problem numerically. One problem arises for higher order elements and nonsmooth Dirichlet data which can cause serious problems in using standard finite element methods (see [25],[23]). Another difficulty lies in the fact that the Dirichlet boundary conditions do not enter the bilinear form naturally and that causes problems for analyzing the finite element method (see [29], [5], [4], [26] for further discussion).

One faces another challenge in the presence of layers which are the regions where the gradient of the solution is large. Usually, the boundary layers occur because of the fact that reduced problem is first order PDEs and requires boundary conditions on inflow part of the boundary only. In this case, standard Galerkin methods fail when $h|\vec{\beta}| > 1$, where h is mesh size, producing highly oscillatory solutions. A lot of research has been done in last 40 years to address this difficulty (see [6], [21], [25], [10], and [30]).

We have an example to illustrate this difficulty in the following simple example,

Example:

$$-0.0025y''(x) + y'(x) = 1, \quad x \in (0, 1),$$

$$y(0) = y(1) = 0.$$

The figure 1.2.1 shows nonphysical oscillations of the standard Galerkin solution for $h = 0.1$.

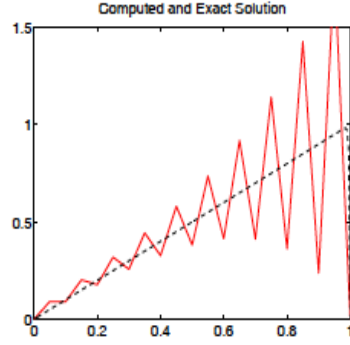


FIGURE 1.2.1: Standard Galerkin

One way to solve this problem is to use stabilized methods (see [34]). We will mention some of them. One of the first stable method of arbitrary order is SUPG (Streamline Upwind Petrov Galerkin) [21], [17], [11]. In this method, the space of test function is different from the space of trial function and chosen such that the method is stable and consistent. Other stabilized methods where the space of trial and test functions are the same and use upwind stabilization are HDG (Hybridizable Discontinuous Galerkin), SIPG (Symmetric Interior Petrov Galerkin) [5], [23], [26], LDG (Local Discontinuous Galerkin) [33], [15]. Another popular stabilized method where the space of trial and test functions are the same is the edge stabilization [12],[7].

DG methods are shown to be robust for the advection-diffusion-reaction problem (see [5], [16]) even for advection-dominated case. DG methods were not only analyzed for the advection-diffusion-reaction problem but also for the optimal control problem of the

advection-diffusion-reaction equation (see [7], [18],[24]). In addition to being stable, the discontinuous Galerkin methods, such as SIPG, usually treat the boundary conditions weakly. SIPG method was also analyzed for distributed optimal control problems and optimal local and global error estimates were obtained (see [27]) but not for the boundary control problems. We would like to investigate the performance of SIPG method applied to Dirichlet boundary control problem (1.2.1 and 1.2.2a-1.2.2b) and prove a priori error estimates. We would also like to perform a number of numerical experiments to confirm our theoretical result which is the main subject of the current work.

In this thesis, we analyze the SIPG solution of the Dirichlet boundary control problem and the difficulties with dealing with the stability issues as well as with the difficulty of the treatment of Dirichlet boundary conditions. This method has some attractive features and offers some advantages. This method is stable and accurate, can be of arbitrary order and has been shown analytically that the boundary layers do not pollute the solution into the subdomain of smoothness [23]. Another attractive feature of the method is that the Dirichlet boundary conditions are enforced weakly through the penalty term and not through the finite dimensional subspace [12]. As a result of the weak treatment of the boundary conditions, the Dirichlet boundary control enters naturally into the bilinear form and makes analysis more natural [29], [5],[4], [14]. Finally, SIPG method has the property that two strategies *Optimize-then-Discretize* and *Discretize-then-Optimize* produce the same discrete optimality system (see [26], [15]), which is not the case for other stabilized methods, for example, SUPG method (see [17]).

Let us show some features of SIPG method with the figure 1.2.2 in the following example.

Example:

$$10^{-9}y''(x) + y'(x) = f(x) \quad \text{on } (0, 1),$$

$$y(0) = y(1) = 0.$$

Figure 1.2.2 shows the behavior of the SIPG solution for $h = 0.1$ and $\epsilon = 1e - 9$. As one can see the solution is stable. The Dirichlet boundary condition at $x = 1$ is almost ignored by the method (compare to SUPG method, figure as a result of weak treatment).

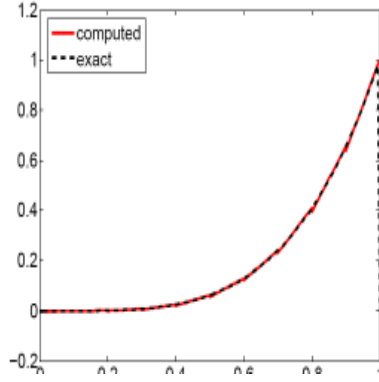


FIGURE 1.2.2: SIPG method

Our choice of this particular DG method was motivated by good approximation and stabilization properties of the method. Additional attractive feature of the method is the weak treatment of the boundary conditions which allows us to set the Dirichlet optimal control problem in natural the finite element frame work and to prove optimal convergence rates for on general convex polygonal domain. Moreover, we state the main result of the thesis is valid for any general convex domain, there exists a positive constant C independent of h such that

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \leq Ch^{1/2}(|\bar{q}|_{H^{1/2}(\Gamma)} + \|\bar{y}\|_{H^1(\Omega)} + \|\hat{y}\|_{L^2(\Omega)}),$$

for h small enough.

The outline of the thesis is as follows. In chapter 2, we set the optimal control problem and show the existence and uniqueness of the problem. In chapter 3, we establish the optimality conditions and deduce the regularity of the optimal solution. In chapter 4, we introduce some basic concepts used in the finite element methods to understand and provide the background for the rest of the thesis. In chapter 5, we give some fundamental definitions to discretize the optimal control problem by using SIPG method and drive the first order discrete optimal system. In chapter 6, we give some auxiliary estimates to use in the main result and analyze the convergence of the solution, then we prove the main result where is given in Theorem 6.1.8. Finally in chapter 7, we performed several numerical examples to support our theoretical results, and additionally when we investigate numerically performance of the method in advection-dominated case.

1.3 Notations

Throughout the thesis we will use the following notations.

- We will use the standard notation for Lebesgue and Sobolev spaces, their suitable norms, and L^2 - inner product. Thus, $(u, v)_\Omega = \int_\Omega uv dx$ and $\langle u, v \rangle_\Gamma = \int_\Gamma uv ds$ are the inner product on the domain Ω and its boundary Γ . The corresponding norms are $\|u\|_{L^2(\Omega)} = (\int_\Omega |u|^2 dx)^{1/2}$ and $\|u\|_{L^2(\Gamma)} = (\int_\Gamma |u|^2 ds)^{1/2}$, respectively.
- $H^{1/2}(\Gamma) = \{u \in L^2(\Gamma) \mid \exists \tilde{u} \in H^1(\Omega) : u = tr(\tilde{u})\}$.
- $\|u\|_{H^{1/2}(\Gamma)} = inf\{\|\tilde{u}\|_{H^1(\Omega)} \mid tr(\tilde{u}) = u\}$.
- $|u|_{H^{1/2}(\Gamma)} = inf\{|\tilde{u}|_{H^1(\Omega)} \mid tr(\tilde{u}) = u\}$.

Chapter 2

Elliptic equations with Dirichlet boundary conditions

2.1 Setting the problem

First, let us consider the state equation,

$$\begin{aligned} -\Delta y + \vec{\beta} \cdot \nabla y + cy &= f \quad \text{in } \Omega, \\ y &= q \quad \text{on } \Gamma. \end{aligned} \tag{2.1.1}$$

We review some regularity results for various conditions on data which we will use later in the analysis. The first result is standard and can be found in many books on partial differential equations, for example [20].

Theorem 2.1.1. *Let $f \in H^{-1}(\Omega)$ and $q \in H^{1/2}(\Gamma)$. Then equation (2.1.1) admits a unique*

solution $y \in H^1(\Omega)$. Moreover, the following estimate holds

$$\|y\|_{H^1(\Omega)} \leq C \left(\|f\|_{H^{-1}(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma)} \right).$$

In the case of $q = 0$ on Γ , $f \in L^2(\Omega)$, and convex Ω , we can obtain a higher regularity of the solution (see [22]).

Theorem 2.1.2. *Let $f \in L^2(\Omega)$ and $q = 0$ on Γ . Then, the equation (2.1.1) admits a unique solution $y \in H^2(\Omega)$ and the following estimate holds*

$$\|y\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Remark: Since the adjoint equation defined by

$$\begin{aligned} -\Delta z - \nabla \cdot (\vec{\beta} z) + cz &= f \quad \text{in } \Omega \\ z &= 0 \quad \text{on } \Gamma, \end{aligned}$$

it is also an advection-diffusion equation and the results of the above theorems are valid for the adjoint equation as well.

The theory in the case of $q \in L^2(\Gamma)$ is more technical and to obtain the desired regularity result, we use *the transposition method* [28], which we will briefly describe next.

The transposition method

Suppose q is smooth enough, $\phi \in L^2(\Omega)$ and let y_1 and y_2 be the solutions of the following equations,

$$\begin{aligned} -\Delta y_1 + \vec{\beta} \cdot \nabla y_1 + cy_1 &= 0 \quad \text{in } \Omega, \\ y_1 &= q \quad \text{on } \Gamma, \end{aligned}$$

and

$$-\Delta y_2 - \nabla \cdot (\vec{\beta} y_2) + c y_2 = \phi \quad \text{in } \Omega,$$

$$y_2 = 0 \quad \text{on } \Gamma,$$

respectively. Then, by the integration by parts and using the fact that $y_2 = 0$ on Γ , we obtain

$$\begin{aligned} 0 &= (-\Delta y_1 + \vec{\beta} \cdot \nabla y_1 + c y_1, y_2)_\Omega \\ &= (\nabla y_1, \nabla y_2)_\Omega - \left(\frac{\partial y_1}{\partial n}, y_2\right)_\Gamma + (y_1 \vec{\beta} \cdot \vec{n}, y_2)_\Gamma - (y_1, \nabla \cdot (\vec{\beta} y_2))_\Omega + (c y_1, y_2)_\Omega \\ &= (\nabla y_1, \nabla y_2)_\Omega - (y_1, \nabla \cdot (\vec{\beta} y_2))_\Omega + (y_1, c y_2)_\Omega \\ &= (y_1, -\Delta y_2)_\Omega + \left(y_1, \frac{\partial y_2}{\partial n}\right)_\Gamma - (y_1, \nabla \cdot (\vec{\beta} y_2))_\Omega + (y_1, c y_2)_\Omega \\ &= (y_1, -\Delta y_2 - \nabla \cdot (\vec{\beta} y_2) + c y_2)_\Omega + \left(y_1, \frac{\partial y_2}{\partial n}\right)_\Gamma \\ &= (y_1, \phi)_\Omega + \left(q, \frac{\partial y_2}{\partial n}\right)_\Gamma, \end{aligned}$$

where $-\Delta y_2 - \nabla \cdot (\vec{\beta} y_2) + c y_2 = \phi$ in Ω and $y_1 = q$ on Γ are used in the last step. Hence we obtain

$$(y_1, \phi)_\Omega = -\left(q, \frac{\partial y_2}{\partial n}\right)_\Gamma.$$

The above formula defines a mapping $\Lambda : \phi \rightarrow -\frac{\partial y_2}{\partial n}$ that is linear and continuous from $L^2(\Omega)$ to $H^{1/2}(\Gamma)$. Since the embedding $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact, Λ is a compact operator from $L^2(\Omega)$ to $L^2(\Gamma)$. Hence, its adjoint Λ^* is a compact operator from $L^2(\Gamma)$ to $L^2(\Omega)$.

Since $(y_1, \phi)_\Omega = -\int_\Gamma q \frac{\partial y_2}{\partial n} = \langle q, \Lambda \phi \rangle_{L^2(\Gamma)}$ and $\langle q, \Lambda \phi \rangle_{L^2(\Gamma)} = \langle \Lambda^* q, \phi \rangle_\Omega$, we conclude that $y_1 = \Lambda^* q$. Using the above, we can define an "ultra-weak" solution for (2.1.1) for Dirichlet data in $L^2(\Gamma)$ as follows.

We say that $y \in L^2(\Omega)$ is a unique ultra-weak solution of the equation (2.1.1) if

$$\int_{\Omega} y\phi = (f, p)_{\langle H^{-1}(\Omega), H_0^1(\Omega) \rangle} - \int_{\Gamma} q \frac{\partial p}{\partial n}, \quad \forall \phi \in L^2(\Omega),$$

where p satisfies

$$\begin{aligned} -\Delta p - \nabla \cdot (\vec{\beta}p) + cp &= \phi \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Now we are ready to provide the following regularity result.

Theorem 2.1.3. *For any $f \in H^{-1}(\Omega)$ and $q \in L^2(\Gamma)$, the problem (2.1.1) admits a unique ultra-weak solution $y \in L^2(\Omega)$. Moreover, the following estimate holds,*

$$\|y\|_{L^2(\Omega)} \leq C(\|f\|_{H^{-1}(\Omega)} + \|q\|_{L^2(\Gamma)}). \quad (2.1.2)$$

Proof. Existence follows from Definition 2.1. For the uniqueness, we assume that y_1 and y_2 are distinct solutions of the problem (2.1.1) and let $u = y_1 - y_2$, then

$$\begin{aligned} -\Delta u - \nabla \cdot (\vec{\beta}u) + cu &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Since $H^1(\Omega)$ is dense in $L^2(\Omega)$, it is enough to consider $u \in H^1(\Omega)$. By Theorem 2.1.1, we have

$$\|u\|_{H^1(\Omega)} = 0.$$

As a result $u = 0$, hence $y_1 = y_2$ and this contradiction proves the uniqueness.

To show the desired estimate (2.1.2) we use a duality argument. Let w be the solution of the problem

$$\begin{aligned} -\Delta w - \nabla \cdot (\vec{\beta}w) + cw &= y \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \Gamma. \end{aligned}$$

By using the above duality argument and using integration by parts and the fact that $w = 0$ on Γ , we obtain

$$\begin{aligned} \|y\|_{L^2(\Omega)}^2 &= (y, -\Delta w - \nabla \cdot (\vec{\beta}w) + cw)_\Omega \\ &= (\nabla y, \nabla w)_\Omega - (y, \frac{\partial w}{\partial n})_\Gamma - (y, w(\vec{\beta} \cdot \vec{n}))_\Gamma + (\vec{\beta} \cdot \nabla y, w)_\Omega + (y, cw)_\Omega \\ &= (-\Delta y, w)_\Omega + (\frac{\partial y}{\partial n}, w)_\Gamma - (y, \frac{\partial w}{\partial n})_\Gamma - (y, w(\vec{\beta} \cdot \vec{n}))_\Gamma + (\vec{\beta} \cdot \nabla y, w)_\Omega + (y, cw)_\Omega \\ &= (-\Delta y + \vec{\beta} \cdot \nabla y + cy, w)_\Omega - (y, \frac{\partial w}{\partial n})_\Gamma \\ &= (f, w)_\Omega - (q, \frac{\partial w}{\partial n})_\Gamma, \end{aligned}$$

where in the last step we use $-\Delta y + \vec{\beta} \cdot \nabla y + cy = f$.

By the trace and the Cauchy-Schwarz inequalities, and by using Theorem 2.1.2 we have the following estimate

$$\begin{aligned} \|y\|_{L^2(\Omega)}^2 &\leq \|f\|_{H^{-1}(\Omega)} \|w\|_{H^1(\Omega)} + \|q\|_{L^2(\Gamma)} \|\frac{\partial w}{\partial n}\|_{L^2(\Gamma)} \\ &\leq C(\|f\|_{H^{-1}(\Omega)} + \|q\|_{L^2(\Gamma)}) \|w\|_{H^2(\Omega)} \\ &\leq C(\|f\|_{H^{-1}(\Omega)} + \|q\|_{L^2(\Gamma)}) \|y\|_{L^2(\Omega)}. \end{aligned}$$

Canceling $\|y\|_{L^2(\Omega)}$ on both sides, we prove the desired estimate (2.1.2).

□

Chapter 3

First order optimality system and the regularity of the optimal solution

Next we will provide the first order optimality conditions for the problem (1.2.1).

Theorem 3.0.4. Assume that $f \in L^2(\Omega)$ and let (\bar{y}, \bar{q}) be the optimal solution of the problem (2.1.1). Then, the optimal control \bar{q} is given by $\alpha \bar{q} = \frac{\partial \bar{z}}{\partial n}$, where \bar{z} is the unique solution of the equation,

$$\begin{aligned} -\Delta \bar{z} - \nabla \cdot (\vec{\beta} \bar{z}) + c \bar{z} &= \bar{y} - \hat{y} \quad \text{in } \Omega, \\ \bar{z} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{3.0.1}$$

Proof. Let (\bar{y}, \bar{q}) be an optimal solution of the equation (1.2.1). We set $F(q) = J(y(q), q)$, where $y(q)$ is the solution of the equation (2.1.1) for a given $q \in L^2(\Gamma)$. Let y_q be the

solution of the problem

$$\begin{aligned} -\Delta y_q + \vec{\beta} \cdot \nabla y_q + c y_q &= f \quad \text{in } \Omega, \\ y_q &= q + \bar{q} \quad \text{on } \Gamma. \end{aligned}$$

By the optimality of (\bar{y}, \bar{q}) , we have that $\frac{1}{\lambda}(F(\bar{q} + \lambda q) - F(\bar{q})) \geq 0$ for all q and $\lambda > 0$, so $F(\bar{q} + q) - F(\bar{q}) \geq 0$.

Equivalently, if $F(\bar{q} + q) - F(\bar{q}) \geq 0$ for all q in $L^2(\Gamma)$, then \bar{q} is an optimal solution of the problem. We find

$$\begin{aligned} F(\bar{q} + q) - F(\bar{q}) &= J(y_q, q + \bar{q}) - J(\bar{y}, \bar{q}) \\ &= \frac{1}{2} \int_{\Omega} (y_q - \bar{y})(y_q + \bar{y} - 2\hat{y}) + \frac{\alpha}{2} \int_{\Gamma} (2q\bar{q} + q^2) \\ &= \frac{1}{2} \int_{\Omega} (y_q - \bar{y})^2 + \frac{\alpha}{2} \int_{\Gamma} q^2 + \int_{\Omega} (y_q - \bar{y})(\bar{y} - \hat{y}) + \alpha \int_{\Gamma} q\bar{q}. \end{aligned}$$

Let \bar{z} be the solution of the equation (3.0.1). Then, we can estimate the third term of the right hand side by using the Green's formula and using the fact that $y_q = \bar{q} + q$ and $\bar{z} = 0$ on Γ . Thus, we obtain

$$\begin{aligned} \int_{\Omega} (y_q - \bar{y})(\bar{y} - \hat{y}) &= \int_{\Omega} (y_q - \bar{y})(-\Delta \bar{z} - \nabla \cdot (\vec{\beta} \bar{z}) + c \bar{z}) \\ &= - \int_{\Gamma} \frac{\partial \bar{z}}{\partial n} (y_q - \bar{q}) + \int_{\Omega} \nabla \bar{z} \cdot \nabla (y_q - \bar{y}) - \int_{\Gamma} (\bar{y} - y_q) \bar{z} (\vec{\beta} \cdot \vec{n}) + \int_{\Omega} \bar{z} (\vec{\beta} \cdot \nabla (\bar{y} - y_q)) \\ &\quad + \int_{\Omega} (\bar{y} - y_q) c \bar{z} \\ &= - \int_{\Gamma} \frac{\partial \bar{z}}{\partial n} (\bar{q} + q - \bar{q}) + \int_{\Omega} \nabla \bar{z} \cdot \nabla (y_q - \bar{y}) + \int_{\Omega} \bar{z} (\vec{\beta} \cdot \nabla (\bar{y} - y_q)) + \int_{\Omega} (\bar{y} - y_q) c \bar{z} \\ &= - \int_{\Gamma} q \frac{\partial \bar{z}}{\partial n} + \underbrace{\int_{\Gamma} \left(\frac{\partial y_q}{\partial n} - \frac{\partial \bar{y}}{\partial n} \right) \bar{z}}_{=0} - \underbrace{\int_{\Omega} \bar{z} (-\Delta (\bar{y} - y_q) + \vec{\beta} \cdot \nabla (\bar{y} - y_q) + c (\bar{y} - y_q))}_{=0}. \end{aligned}$$

By setting $\alpha \bar{q} = \frac{\partial \bar{z}}{\partial n}$, we have

$$\int_{\Omega} (y_q - \bar{y})(\bar{y} - \hat{y}) = - \int_{\Gamma} q \frac{\partial \bar{z}}{\partial n} = -\alpha \int_{\Gamma} q \bar{q}.$$

Putting all results together, we have

$$\begin{aligned} F(\bar{q} + q) - F(\bar{q}) &= \frac{1}{2} \int_{\Omega} (y_q - \bar{y})^2 + \frac{\alpha}{2} \int_{\Gamma} q^2 - \alpha \int_{\Gamma} q \bar{q} + \alpha \int_{\Gamma} q \bar{q} \\ &= \frac{1}{2} \int_{\Omega} (y_q - \bar{y})^2 + \frac{\alpha}{2} \int_{\Gamma} q^2 \geq 0, \end{aligned}$$

i.e. (\bar{y}, \bar{q}) is the optimal solution to the problem (3.0.1) with $\bar{q} = \frac{1}{\alpha} \frac{\partial \bar{z}}{\partial n}$. □

3.0.1 Strong Form of the First Order Optimality Conditions

The first order optimality conditions in the strong form are as the following

$$\text{Adjoint equation} \begin{cases} -\Delta z - \vec{\beta} \cdot \nabla z + (c - \nabla \cdot \vec{\beta})z = \hat{y} - y & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases} \quad (3.0.2)$$

$$\text{Gradient equation} \begin{cases} \frac{\partial z}{\partial n} = \alpha q & \text{on } \Gamma. \end{cases} \quad (3.0.3)$$

$$\text{State equation} \begin{cases} -\Delta y + \vec{\beta} \cdot \nabla y + cy = f & \text{in } \Omega, \\ y = q & \text{on } \Gamma. \end{cases} \quad (3.0.4)$$

3.0.2 Regularity

In next theorem, we establish the regularity of the optimal solution of the problem (1.2.2a) and (1.2.2b).

Theorem 3.0.5. *Let $(\bar{y}, \bar{q}) \in L^2(\Omega) \times L^2(\Gamma)$ be the optimal solution to the optimization problem (1.2.1) subject to the problem (1.2.2a) and (1.2.2b), and \bar{z} be the optimal adjoint state. Then,*

$$(\bar{y}, \bar{q}, \bar{z}) \in H^1(\Omega) \times H^{1/2}(\Gamma) \times H^2(\Omega).$$

Proof. For $\bar{q} \in L^2(\Gamma)$, from the state equation (3.0.4), $\bar{y} \in L^2(\Omega)$ holds by Theorem 2.1.3. Since $\bar{y}, \hat{y} \in L^2(\Omega)$ and Ω is a convex domain, from the adjoint equation 3.0.2, $\bar{z} \in H^2(\Omega)$ holds by Theorem 2.1.2.

Since $\bar{z} \in H^2(\Omega)$, we have $\frac{\partial \bar{z}}{\partial n} \in H^{1/2}(\Gamma)$, from the gradient equation (3.0.3), $\alpha \bar{q} = \frac{\partial \bar{z}}{\partial n}$ implies $\bar{q} \in H^{1/2}(\Gamma)$.

Since $\bar{q} \in H^{1/2}(\Gamma)$, from the state equation (3.0.4), $\bar{y} \in H^1(\Omega)$ holds by Theorem 2.1.1.

□

Remark: Using regularity results from [31], we can generalize the regularity which depends on the largest interior angle of the polygonal domain w in \mathbb{R}^2 .

Let $\lambda = \pi/w \in (\frac{1}{2}, 3]$ be the leading singularity exponent. Then, the regularity is generalized as the following

$$(\bar{y}, \bar{q}, \bar{z}) \in H^{\min(\lambda, 2) - \epsilon}(\Omega) \times H^{\min(\lambda - \frac{1}{2}, \frac{3}{2})}(\Gamma) \times H^{\min(1 + \lambda - \epsilon, 3)}(\Omega).$$

Chapter 4

Basic Concepts for the Finite Element Methods

The Finite Element Methods (FEM) are widely used computational methods in science and engineering. The goal of the methods is to approximate the solution of the problem from the infinite dimensional space (solution space) by a linear combination of functions from the finite dimensional space (trial space) called *trial functions*. Usually, the space of trial functions consists of piecewise polynomial functions defined on "elements" that partition the given bounded domain. Let us explain that concisely and some details and definitions are given in the text books by Brenner and Scott [9] and Braess [8].

4.1 The finite element method

Usually, the starting point of the FEM is the variational equation. Let the problem be to find $v \in V$ such that

$$a(u, v) = \ell(v) \quad \forall v \in V, \quad (4.1.1)$$

where V in a Hilbert space .

The main tool that guarantees the existence and uniqueness of the equation above is the *Lax-Milgram Lemma* as in the following [19].

Lemma 4.1.1 (Lax-Milgram). *Let V be in a Hilbert space, and $a(\cdot, \cdot)$ be a bilinear form on $(V \times V)$ and $\ell(\cdot)$ be a linear form on V with corresponding norms $\|\cdot\|_V$, $\|\cdot\|_{V'}$, respectively. Assume that the followings are hold:*

- $a(\cdot, \cdot)$ is continuous, i.e. $|a(u, v)| \leq C\|u\|_V\|v\|_V$,
- $a(\cdot, \cdot)$ is coercive, i.e. $a(u, u) \geq \alpha\|u\|_V^2$ for some $\alpha > 0$, $\forall u \in V$,
- $\ell(\cdot)$ is continuous, i.e. $|\ell(v)| \leq \gamma\|v\|_V$ for some $\gamma > 0$.

Then, there exists a unique solution $u \in V$ of the problem 4.1.1. Moreover, we have a priori estimate

$$\|u\|_V \leq \frac{1}{\alpha}\|\ell\|_{V'}.$$

The idea of the FEM is to construct V_h defined on a finite dimensional space that is well approximate V . The Galerkin FEM is to find $u_h \in V_h$ such that

$$a_h(u_h, v) = \ell_h(v) \quad \forall v \in V_h, \quad (4.1.2)$$

where V_h is a finite dimensional space and h is a discretization parameter .

We can easily see that if $a_h(\cdot, \cdot)$ satisfies the conditions of *Lax-Milgram Lemma*, the problem 4.1.2 has a unique solution for each h .

Moreover, if we express u_h in terms of a linear combination of basis functions, then we

observe that 4.1.2 is equivalent to a square system of linear equations of the form

$$KU = F$$

where K is matrix and F is vector.

As a consequence of Lax-Milgram lemma, K is nonsingular, and u_h is a good approximation to u i.e. $u_h \rightarrow u$ as $\dim(V_h) \rightarrow \infty$. Here, the main problem is to choose trial space V_h for the desired approximation, so we would like to follow these steps for the intended choice:

- V_h should approximate V well.
- Basis function should be simple enough to generate the matrix K and vector F in the matrix form $KU = F$ of the method.
- Solve $KU = F$ efficiently.

Abstract of the FEM formulation

Definition 4.1.2. Let

- (i) $K \subseteq \mathbb{R}^n$ be a domain with piecewise smooth boundary (the **element domain**)
- (ii) P be a finite dimensional space of functions on K (the **shape function**)
- (iii) $N = \{N_1, N_2, \dots, N_k\}$ be a basis for P' (the **nodal variables**)

Then, (K, P, N) is called a **finite element**.

Definition 4.1.3. Let (K, P, N) be a finite element, and let $\{\phi_1, \phi_2, \dots, \phi_k\}$ be the basis for P dual to N ($N_i(\phi_j) = \delta_{ij}$). It is called the **nodal basis** for P .

When explaining steps, the FEM is to find the approximate solution u_h by linear combination of local polynomials into the finite elements, which is easily differentiable and

integrable and has local supports on each element, so it is a good approximation while partition goes to zero. Thus, $u_h = \sum_{j=1}^d N_j \phi_j$ where $d = \dim(V_h)$, leads to the discrete problem 4.1.2 in the following form:

$$\sum_{j=1}^d N_j a_h(\phi_j, \phi_i) = \ell_h(f, \phi_i) \quad 1 \leq i \leq d.$$

Thus, it is equivalent to solving matrix equation

$$KU = F,$$

where $K = a_h(\phi_j, \phi_i)$ is a "stiffness" matrix, $F = \ell_h(\phi_i)$ is a vector and $U = N_i$ is the coordinate vector of u_h for $1 \leq i \leq \dim(V_h)$. This assembly process leads to the system of equations that we want to solve.

Now, we can evaluate the error of the convergence of V_h to V , but before estimating, we consider two cases about V_h which has the conforming finite elements (*Lagrange elements*) and the nonconforming finite elements:

1. The Conforming FE (" $V_h \subset V$ ")

We have a simple example to illustrate this case and let us have the following second order elliptic equation.

Example:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Thus, the variational form is $a_h(y, v) = \ell_h(v)$, where

$$a_h(y, v) = \sum_{\tau \in T_h} (\nabla y, \nabla v)_\tau = (\nabla y, \nabla v)_\Omega$$

and

$$\ell_h(v) = \sum_{\tau \in T_h} (f, v)_\tau = (f, v)_\Omega.$$

τ and T_h in the form above are used as an each element and the triangulation of the domain Ω , respectively. For this example, the solution space $V = H_0^1(\Omega)$. It is known that $V_h \subset C^0(\bar{\Omega})$ where $\bar{\Omega} = \cup_{\tau_i \in T_h} \tau_i$ if and only if for all functions $v \in V_h$ should be continuous across the common boundary of neighboring elements, then $V_h \subset V$.

In this case, as a result of the conformality, the test space and solution space are identical by choosing

$$a_h(\cdot, \cdot) = a(\cdot, \cdot)$$

and

$$\ell_h(\cdot) = \ell(\cdot).$$

Thus, 4.1.2 is replaced by the following discrete problem which is to find $u_h \in V_h$ such that

$$a(u_h, v) = \ell(f, v) \tag{4.1.3}$$

for all $v \in V_h$.

As a result, the solution has *Galerkin Orthogonality* property.

Galerkin Orthogonality

Let u and u_h be the solution of the problems 4.1.1 and 4.1.3, respectively. Then,

$$a(u - u_h, v) = 0 \quad \forall v \in V_h.$$

Now, we can estimate the error of the approximation.

Theorem 4.1.4. (Cea's Theorem)

Let u and u_h be the solution of the problems 4.1.1 and 4.1.3, respectively. Then, we have

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{\{v \in V_h\}} \|u - v\|_V,$$

where C is the continuity constant and α is the coersivity constant of $a(., .)$ on V .

Proof. For all $v \in V_h$, by using coersivity and continuity,

$$\begin{aligned} \alpha \|u - u_h\|_V^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v) + \underbrace{a(u - u_h, v - u_h)}_0 \\ &= a(u - u_h, u - v) \\ &\leq C \|u - u_h\|_V \|u - v\|_V, \end{aligned}$$

where we use the fact that $v - u_h \in V_h$ and Galerkin orthogonality in the last step.

Thus,

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \|u - v\|_V$$

for all $v \in V_h$. Since V_h is closed, we conclude the desired result. \square

2. The Nonconforming FE (" $V_h \not\subset V$ ")

Let us explain this case with the following example.

Example:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Similarly, $V_h \subset H_0^1(\Omega)$ if and only if the functions $v \in V_h$ should be continuous across the common boundary of neighboring elements. However, if v is discontinuous piecewise polynomials, v has jumps across the elements, then $v \notin C^0(\bar{\Omega})$ i.e. $v \notin H^1(\Omega)$, so " $V_h \not\subset V$ ". Whereas, we can define the FEM and provide the convergence by using the discontinuous Galerkin method which is defined on polynomials on each element τ such that

$$V_h = \{v \in L^2(\Omega) : v|_{\tau} \in \wp^k(\tau) \quad \forall \tau \in T_h\},$$

where \wp^k is the space of polynomials of degree at most k on each element τ or edge E_h with no continuity requirement.

Thus, to set the DG method, we define a bilinear form

$$a_h(y, v) = \sum_{\tau \in T_h} (\nabla y, \nabla v)_{\tau} + \sum_{e \in E_h} \left[\frac{\gamma}{h} ([y], [v])_e - (\{\nabla y\}, [v])_e - ([y], \{\nabla v\})_e \right]$$

and

$$\ell_h(v) = \sum_{\tau \in T_h} (f, v)_{\tau}$$

and mesh dependent norm

$$\|v\|_h^2 = \sum_{\tau \in T_h} \|\nabla v\|_{\tau}^2 + \|v\|_{\tau}^2 + \sum_{e \in E_h^{\partial}} \frac{\gamma}{h} \| [v] \|_e^2,$$

then DG solution is defined as a solution of $a_h(y, v) = \ell_h(v)$ for all $v \in V_h$. For details

and notations, see chapter 5.

In this case, we have $a_h(\cdot, \cdot) \neq a(\cdot, \cdot)$ as clearly seen and the Galerkin orthogonality is not satisfied. However, we still have error estimate given by *Strang's second lemma*.

Lemma 4.1.5. (Strang's second Lemma)

Assume that the variational form 4.1.2 is defined on $(V_h + V) \times V_h$ and $\ell(\cdot)$ is a linear form on V_h with mesh dependent norm $\|\cdot\|_h$. If the followings are hold:

- $a_h(\cdot, \cdot)$ is continuous, i.e. $|a_h(u, v)| \leq C\|u\|_h\|v\|_h$, $u \in (V_h + V)$ and $v \in V_h$.
- $a_h(\cdot, \cdot)$ is coercive, i.e. $a_h(u, u) \geq \alpha\|u\|_h^2$, for some $\alpha > 0$, $\forall u \in (V_h)$.
- $\ell_h(\cdot)$ is continuous. Then, the following estimate holds:

$$\|u - u_h\|_h \leq C \left(\inf_{v \in V_h} (\|u - v\|_h) + \sup_{v_s \in V_h} \frac{|a_h(u, v_s) - \ell_h(v_s)|}{\|v_s\|_h} \right).$$

Proof. Let $v \in V_h$ be an arbitrary element. From the triangle inequality,

$$\|u - u_h\|_h \leq \|u - v\|_h + \|u_h - v\|_h$$

For convenience, set $v_s = u_h - v$. By coersivity, boundedness and Cea's lemma,

$$\begin{aligned} \alpha\|u_h - v\|_h^2 &\leq a_h(u_h - v, v_s) = a_h(u_h - u, v_s) + a_h(u - v, v_s) \\ &= \ell_h(v_s) - a_h(u, v_s) + a_h(u - v, v_s) \\ &\leq C(\|u - v\|_h\|v_s\|_h + |\ell_h(v_s) - a_h(u, v_s)|) \end{aligned}$$

By dividing $\|v_s\|_h$ and using continuity of $a_h(\cdot, \cdot)$, we obtain

$$\|u_h - v\|_h \leq C(\|u - v\|_h + \frac{|a_h(u, v_s) - \ell_h(v_s)|}{\|v_s\|_h}).$$

Then, from the above triangle inequality we conclude

$$\|u - u_h\|_h \leq C \left(\inf_{v \in V_h} (\|u - v\|_h + \sup_{v_s \in V_h} \frac{|a_h(u, v_s) - \ell_h(v_s)|}{\|v_s\|_h}) \right).$$

□

Remark: it can be seen that if $a_h(\cdot, \cdot) = a(\cdot, \cdot)$ and $\ell_h(\cdot) = \ell(\cdot)$, the last term on the right hand side in the estimate vanishes, so we obtain Cea's lemma. The first term on the right hand side is called as *the approximation error* and the second term as *consistency error*.

Chapter 5

Discontinuous Galerkin Discretization

We consider a family of conforming quasi-uniform shape regular triangulations T_h of Ω such that $\bar{\Omega} = \cup_{\tau_i \in T_h} \tau_i$ and $\tau_i \cap \tau_j = \emptyset \ \forall \tau_i, \tau_j \in T_h, \ i \neq j$ with a mesh size

$$h = \sup_{\tau_i \in T_h} \text{diam}(\tau_i).$$

We define E_h as a collection of all edges $E_h = E_h^0 \cup E_h^\partial$ where E_h^0 and E_h^∂ are the collections of interior and boundary edges, respectively, and we decompose the boundary edges as $E_h^\partial = E_h^+ \cup E_h^-$, where

$$E_h^- := \{e \in E_h^\partial : e \subset \{x \in \Gamma : \vec{\beta}(x) \cdot \vec{n}(x) < 0\}\}$$

and $E_h^+ := E_h^\partial \setminus E_h^-$ i.e. these are the collections of the edges that belong to the inflow and outflow part of the boundary, respectively.

We define the standard jumps and averages on the set of interior edges by

$$\begin{aligned}\{\varphi\} &= \frac{\varphi_1 + \varphi_2}{2}, & [[\varphi]] &= \varphi_1 \vec{n}_1 + \varphi_2 \vec{n}_2, \\ \{\vec{\phi}\} &= \frac{\vec{\phi}_1 + \vec{\phi}_2}{2}, & [[\vec{\phi}]] &= \vec{\phi}_1 \cdot \vec{n}_1 + \vec{\phi}_2 \cdot \vec{n}_2,\end{aligned}$$

where \vec{n}_1 and \vec{n}_2 are outward normal vectors at the boundary edge of neighboring elements τ_1 and τ_2 , respectively. Define the discrete state and control spaces as

$$V_h := \{y \in L^2(\Omega) : y|_\tau \in \wp^k \quad \forall \tau \in T_h\},$$

$$Q_h := \{q \in L^2(\Gamma) : q|_\tau \in \wp^l \quad \forall \tau \in E_h^\partial\},$$

respectively. We denote by \wp^k the space of polynomials of degree at most k on each element or edge. In general, the state and control variables can be approximated by polynomials of different degrees $k, l \in \mathbb{N}$.

Here, we use *Symmetric Interior Penalty Galerkin Method (SIPG)* to approximate to the problem. In deriving the SIPG method, we use the following identity

$$\begin{aligned}\sum_{\tau \in T_h} (\vec{\phi} \cdot \vec{n}, \varphi)_{\partial\tau} &= \sum_{e \in E_h} (\{\vec{\phi}\}, [[\varphi]])_e + \sum_{e \in E_h^0} ([[\vec{\phi}]], \{\varphi\})_e \\ &= \sum_{e \in E_h^0} (\{\vec{\phi}\}, [[\varphi]])_e + ([[\vec{\phi}]], \{\varphi\})_e + \sum_{e \in E_h^\partial} (\vec{\phi} \cdot \vec{n}, \varphi)_e.\end{aligned}$$

For a given $q \in Q_h$, the SIPG solution $y \in V_h$ satisfies

$$a_h(y, v) = \ell_h(f; q, v) \quad \forall v \in V_h,$$

where

$$\begin{aligned}
a_h(y, v) &= \sum_{\tau \in T_h} (\nabla y, \nabla v)_\tau \sum_{\tau \in T_h} (\vec{\beta} \cdot \nabla y + cy, v)_\tau \\
&+ \sum_{e \in E_h} \left[\frac{\gamma}{h} ([[y]], [[v]])_e - (\{\nabla y\}, [[v]])_e - ([[y]], \{\nabla v\})_e \right] \\
&+ \sum_{e \in E_h^0} (y^+ - y^-, |\vec{n} \cdot \vec{\beta}| v^+)_e + \sum_{e \in E_h^-} (y^+, v^+ |\vec{n} \cdot \vec{\beta}|)_e,
\end{aligned} \tag{5.0.1}$$

and

$$\ell_h(f; q, v) = \sum_{\tau \in T_h} (f, v)_\tau + \sum_{e \in E_h^\partial} \left(\frac{\gamma}{h} (q, [[v]])_e - (q, \{\nabla v\})_e \right) + \sum_{e \in E_h^-} (q, v^+ |\vec{n} \cdot \vec{\beta}|)_e. \tag{5.0.2}$$

Then, we can define the energy norm as

$$|||v|||^2 = \sum_{\tau \in T_h} \|\nabla v\|_\tau^2 + \|v\|_\tau^2 + \sum_{e \in E_h^\partial} \frac{\gamma}{h} \|[[v]]\|_e^2.$$

5.0.1 Well-Posed

It has been shown, for example [5], that the bilinear form $a_h(\cdot, \cdot)$ is coercive and bounded on V_h i.e. $a_h(v, v) \geq C |||v|||^2$ and $a_h(u, v) \leq C |||u||| |||v|||$, respectively. Thus, *Lax-Milgram Lemma* guarantees the existence of a unique solution $u_h \in V_h$ of the equation

$$a_h(u_h, v_h) = \ell_h(f; q, v_h)$$

for all $v_h \in V_h$. In the following, we will also need to the trace and the inverse inequalities to derive a priori error estimate in the energy norm. There are two approaches which are *Optimize-then-Discretize* and *Discretize-then-Optimize*.

5.0.2 Weak form of the first order optimality conditions (optimize-then-discretize approach)

Theorem 5.0.6. The solution of the optimization problem (1.2.1) is characterized by the Euler-Lagrange principle, stating that the pair $(\bar{y}_h, \bar{q}_h) \in V_h \times Q_h$ is an optimal discrete solution if and only if there exists an "adjoint state" $\bar{z}_h \in V_h$ such that the triplet $(\bar{y}_h, \bar{q}_h, \bar{z}_h) \in V_h \times Q_h \times V_h$ solves the discretized optimality system:

$$a_h(\psi, \bar{z}_h) = \ell_h(\hat{y} - \bar{y}_h; 0, \psi) \quad \forall \psi \in V_h, \quad (5.0.3a)$$

$$\left\langle \frac{\partial \bar{z}_h}{\partial n}, \phi \right\rangle_\Gamma = -\alpha \langle \bar{q}_h, \phi \rangle_\Gamma \quad \forall \phi \in Q_h, \quad (5.0.3b)$$

$$a_h(\bar{y}_h, \varphi) = \ell_h(f; q_h, \varphi) \quad \forall \varphi \in V_h. \quad (5.0.3c)$$

5.0.3 Discrete optimality system (discretize-then-optimize approach)

We apply the SIPG discretization to the optimal control problem (1.2.1). Now, define the discrete Lagrangian as

$$L_h(\bar{y}_h, \bar{q}_h, \bar{z}_h) = J(\bar{y}_h, \bar{q}_h) + a_h(\bar{y}_h, \bar{z}_h) - \ell_h(f, \bar{q}_h).$$

Then, setting the partial Frechet derivatives to be zero, we obtain the discrete optimality system:

$$\frac{\partial L_h}{\partial \bar{y}_h} \psi_h = 0 \quad \forall \psi_h \in V_h, \Rightarrow a_h(\psi_h, \bar{z}_h) = \ell_h(\hat{y} - \bar{y}_h; 0, \psi_h). \quad (5.0.4a)$$

$$\frac{\partial L_h}{\partial \bar{q}_h} \phi_h = 0 \quad \forall \phi_h \in Q_h, \Rightarrow \left\langle \frac{\partial \bar{z}_h}{\partial n}, \phi_h \right\rangle_\Gamma = -\langle \alpha \bar{q}_h, \phi_h \rangle_\Gamma + \frac{\gamma}{h} \langle \bar{z}_h, \phi_h \rangle_\Gamma + \langle \bar{z}_h | \vec{n} \cdot \vec{\beta} |, \phi_h \rangle_\Gamma. \quad (5.0.4b)$$

$$\frac{\partial L_h}{\partial \bar{z}_h} \varphi_h = 0 \quad \forall \varphi_h \in V_h, \Rightarrow a_h(\varphi_h, \bar{y}_h) = \ell_h(f; q, \varphi_h). \quad (5.0.4c)$$

As we can see after comparing two approaches, the additional terms are $\frac{\gamma}{h} \langle \bar{z}_h, \phi_h \rangle_\Gamma$ and $\langle \bar{z}_h | \vec{n} \cdot \vec{\beta} |, \phi_h \rangle_\Gamma$. However, since $z = 0$ on the boundary Γ , we can modify the continuous gradient equation, and then the two approaches come to be equivalent. All details and proofs are given in [26]. Here, we consider the second approach, Discretize-Then-Optimize.

Chapter 6

DG Error Estimates

6.1 Auxiliary Estimates

We will need some auxiliary estimates that we will use in the proof of the main result. First, we have some standard estimates which are trace and inverse inequalities.

Lemma 6.1.1. *There exist positive constants C_{tr} and C_{inv} independent of τ and h such that for $\forall \tau \in T_h$*

$$\|v\|_{\partial\tau} \leq C_{tr}(h^{-1/2}\|v\|_{\tau} + h^{1/2}\|\nabla v\|_{\tau}) \quad \forall v \in H^1(\tau) \quad (6.1.1)$$

$$\|\nabla v_h\|_{\tau} \leq C_{inv}h^{-1}\|v_h\|_{\tau} \quad \forall v_h \in V_h. \quad (6.1.2)$$

Then, we need some basic estimates for L^2 - Projection where $P_h : L^2(\Omega) \rightarrow V_h$ is the orthogonal projection such that $(P_h v, \chi)_{\tau} = (v, \chi)_{\tau}$ for all $v \in L^2(\tau)$ and $\chi \in V_h$.

Lemma 6.1.2. *Let P_h be L^2 - projection. Then, we have*

$$\begin{aligned}\|v - P_h v\|_{L^2(\tau)} &\leq Ch^{k+1} \|v\|_{H^{k+1}(\tau)} \\ \|\nabla(v - P_h v)\|_{L^2(\tau)} &\leq Ch^k \|v\|_{H^{k+1}(\tau)}\end{aligned}$$

for $k = 0, 1, 2, \dots$

Now, we are ready to show the error estimate of SIPG solution in the energy norm.

Lemma 6.1.3.

Let v be the unique solution of the problem (2.1.1) and $v_h \in V_h$ be the SIPG solution. Then,

$$|||v - v_h||| \leq Ch^k \|v\|_{H^{k+1}(\Omega)}$$

for $k = 0, 1, 2, \dots$

Proof. Let $e = v - v_h$, then by using the coersivity, the Galerkin orthogonality and bound-
edness of the bilinear form (5.0.1), we have

$$|||e|||^2 \leq a(e, e) = a(e, v - \chi) \leq C |||e||| |||v - \chi||| \quad \forall \chi \in V_h.$$

Thus, we obtain $|||e||| \leq C |||v - \chi|||$.

Next, choosing $\chi = P_h v$, then we have

$$\begin{aligned}|||e|||^2 &\leq C |||v - P_h v|||^2 \\ &= C \left[\underbrace{\sum_{\tau \in T_h} \|\nabla(v - P_h v)\|_{L^2(\tau)}^2}_I + \underbrace{\|v - P_h v\|_{L^2(\tau)}^2}_{II} + \underbrace{\sum_{e \in E_h^\partial} \frac{\gamma}{h} \|[[v - P_h]]\|_{L^2(e)}^2}_{III} \right].\end{aligned}$$

Estimating each term on the right hand side, then we have

I+II:

By using the approximation Lemma 6.1.2, we have

$$\begin{aligned}
& \sum_{\tau \in T_h} \|\nabla(v - P_h v)\|_{L^2(\tau)}^2 + \sum_{\tau \in T_h} \|(v - P_h v)\|_{L^2(\tau)}^2 \\
& \leq \sum_{\tau \in T_h} C h^{2k} \|v\|_{H^{k+1}(\tau)}^2 + \sum_{\tau \in T_h} C h^{2k+2} \|v\|_{H^{k+1}(\tau)}^2 \\
& \leq C h^{2k} \|v\|_{H^{k+1}(\Omega)}^2 + C h^{2k+2} \|v\|_{H^{k+1}(\Omega)}^2 \\
& \leq C h^{2k} \|v\|_{H^{k+1}(\Omega)}^2.
\end{aligned}$$

III:

By using 6.1.1 in Lemma 6.1.1,

$$\begin{aligned}
& \sum_{e \in E_h^\partial} \frac{\gamma}{h} \| [v - P_h v] \|_e^2 = \sum_{e \in E_h^\partial} \frac{\gamma}{h} \|v - P_h\|_{L^2(e)}^2 \\
& \leq \sum_{\tau \in T_h} C \frac{\gamma}{h} (h^{-1} \|v - P_h v\|_{L^2(\tau)} + h \|\nabla(v - P_h v)\|_{L^2(\tau)}) \\
& \leq \sum_{\tau \in T_h} C \frac{\gamma}{h} (h^{-1} h^{2k+2} \|v\|_{H^{k+1}(\Omega)}^2 + h h^{2k} \|v\|_{H^{k+1}(\Omega)}^2) \\
& \leq C_\gamma h^{2k} \|v\|_{H^{k+1}(\Omega)}^2.
\end{aligned}$$

Combining the above estimates and taking the square root, we conclude

$$|||e||| \leq C h^k \|v\|_{H^{k+1}(\Omega)}.$$

□

Next, we will need the estimate of L^2 - *Projection* on the boundary Γ where

$$P_h^\partial : L^2(\Gamma) \rightarrow Q_h$$

is defined by $\langle q - P_h^\partial q, \phi_h \rangle_e = 0$ for all $\phi_h \in \wp^s(e)$.

Lemma 6.1.4.

Let P_h^∂ be L^2 - projection defined on the boundary. Then, for any edge $e \in E^\partial$

$$\|q - P_h^\partial q\|_{L^2(e)} + h^s \|q - P_h^\partial q\|_{W^{s,p}(e)} \leq h^s |q|_{W^{s,p}(e)} \quad \forall e \in E_h^\partial,$$

where E_h^∂ is the set of boundary edges, $q \in W^{s,p}(e)$, $0 \leq s \leq 1$, and $1 < p < \infty$. [29]

Since SIPG method treats the boundary conditions weakly, SIPG solution is not zero on the boundary even if its continuous solution is. However, the following result says that the norm of SIPG solution on the boundary is rather small.

Lemma 6.1.5.

Let us define auxiliary variable \tilde{z} to be a solution of the problem (3.0.2)

$$\begin{aligned} -\Delta \tilde{z} - \nabla \cdot (\vec{\beta} \tilde{z}) + c \tilde{z} &= \hat{y} - y_h \text{ in } \Omega \\ \tilde{z} &= 0 \quad \text{on } \Gamma, \end{aligned}$$

and $\tilde{z}_h \in V_h$ be the SIPG approximation solution. Then,

$$\|\tilde{z}_h\|_{L^2(\Gamma)} \leq Ch^{3/2} \|\hat{y} - y_h\|_{L^2(\Omega)}.$$

Proof. Let \tilde{z} be a solution to the equation (3.0.2). Since

$$\|\tilde{z}_h\|_{L^2(\Gamma)} = \|\tilde{z}_h - \tilde{z}\|_{L^2(\Gamma)} = \|[\tilde{z}_h - \tilde{z}]\|_{L^2(\Gamma)} \leq Ch^{1/2} \|\tilde{z}_h - \tilde{z}\|,$$

we can use the error estimate in the energy norm. Thus, by Lemma 6.1.3 and Lemma 2.1.2, we have that

$$\|\tilde{z}_h\|_{L^2(\Gamma)} \leq Ch^{1/2} \|\tilde{z}_h - \tilde{z}\| \leq Ch^{1/2} h \|\tilde{z}\|_{H^2(\Omega)} \leq Ch^{3/2} \|\hat{y} - y_h\|_{L^2(\Omega)}.$$

□

The estimate of $\|y - y_h\|$ is more involved since $(y - y_h)$ does not satisfy the Galerkin orthogonality. First, we can show the following result.

Lemma 6.1.6. *Let y and y_h satisfy*

$$a_h(y, v) = \ell_h(f; q, v) \quad \forall v \in V_h,$$

$$a_h(y_h, \chi) = \ell_h(f; q, \chi) \quad \forall \chi \in V_h.$$

Then,

$$\|y - y_h\| \leq C(h^{-1/2} \|q - q_h\|_{L^2(\Gamma)} + \|y\|_{H^1(\Omega)}).$$

Proof. By the coersivity, adding and subtracting $P_h y$, we have

$$\|y - y_h\|^2 \leq a_h(y - y_h, y - y_h) = \underbrace{a_h(y - y_h, P_h y - y_h)}_I + \underbrace{a_h(y - y_h, y - P_h y)}_{II}.$$

II :

By using the boundedness of $a_h(\cdot, \cdot)$ and Lemma 6.1.3 for $k = 0$, we obtain

$$a_h(y - y_h, y - P_h y) \leq |||y - y_h||| |||y - P_h y||| \leq C |||y - y_h||| ||y||_{H^1(\Omega)}.$$

I :

Since $(P_h y - y_h) \in V_h$, we have

$$a_h(y - y_h, P_h y - y_h) = \ell_h(0; q - q_h, P_h y - y_h).$$

Then, we have

$$\begin{aligned} \ell_h(0; q - q_h, P_h y - y_h) &= \sum_{e \in E_h^\partial} \left(\frac{\gamma}{h} (q - q_h, [[P_h y - y_h]])_e - (q - q_h, \{\nabla(P_h y - y_h)\})_e \right) \\ &\quad + \sum_{e \in E_h^-} (q - q_h, (P_h y - y_h)^+ |n \cdot \beta|)_e. \end{aligned}$$

By the definition of $\ell(\cdot)$, we can see that the dominating term is $\sum_e \frac{\gamma}{h} (q - q_h, [[P_h y - y_h]])_e$.

Using the fact that $||[[P_h y - y_h]]||_{L^2(\Gamma)}$ is a part of the energy norm and Lemma 6.1.3 for

$k = 0$, we have

$$\begin{aligned} \ell(0; q - q_h, P_h y - y_h) &\leq C \sum_e \frac{\gamma}{h} (q - q_h, [[P_h y - y_h]])_e \\ &\leq Ch^{-1} \left(\sum_e \|q - q_h\|_{L^2(e)}^2 \right)^{1/2} \left(\sum_e ||[P_h y - y_h]||_{L^2(e)}^2 \right)^{1/2} \\ &\leq Ch^{-1} \|q - q_h\|_{L^2(\Gamma)} ||[P_h y - y_h]||_{L^2(\Gamma)} \\ &\leq Ch^{-1} \|q - q_h\|_{L^2(\Gamma)} h^{1/2} ||P_h y - y_h|| \\ &\leq Ch^{-1/2} \|q - q_h\|_{L^2(\Gamma)} (||P_h y - y_h|| + ||y - y_h||) \\ &\leq Ch^{-1/2} \|q - q_h\|_{L^2(\Gamma)} (||y||_{H^1(\Omega)} + ||y - y_h||). \end{aligned}$$

The other terms in $\ell_h(0; q - q_h, P_h y - y_h)$ can be estimated similarly. Thus,

$$\begin{aligned}
|||y - y_h|||^2 &\leq I + II \\
&\leq C\|y\|_{H^1(\Omega)} |||y - y_h||| \\
&\quad + Ch^{-1/2}\|q - q_h\|_{L^2(\Gamma)} |||y - y_h||| + Ch^{-1/2}\|q - q_h\|_{L^2(\Gamma)} \|y\|_{H^1(\Omega)} \\
&\leq \frac{1}{4} |||y - y_h|||^2 + Ch^{-1}\|q - q_h\|_{L^2(\Gamma)}^2 + C\|y\|_{H^1(\Omega)}^2.
\end{aligned}$$

By first taking the square root and then canceling $|||y - y_h|||$, we obtain

$$|||y - y_h||| \leq C(h^{-1/2}\|q - q_h\|_{L^2(\Gamma)} + \|y\|_{H^1(\Omega)}).$$

□

Using a duality, we can show better estimate in L^2 norm.

Lemma 6.1.7.

Let y be the solution of the problem (2.1.1) and y_h in V_h satisfy the bilinear form 5.0.1.

Then,

$$\|y - y_h\|_{L^2(\Omega)} \leq C(h^{1/2}\|q - q_h\|_{L^2(\Gamma)} + h\|y\|_{H^1(\Omega)}).$$

Proof. Since y_h is not a Galerkin projection of y , let us define \tilde{y}_h by $a_h(y - \tilde{y}_h, \chi) = 0$ for $\chi \in V_h$. Then, by the triangle inequality, we have

$$\|y - y_h\|_{L^2(\Omega)}^2 \leq \underbrace{\|y - \tilde{y}_h\|_{L^2(\Omega)}^2}_{K_1} + \underbrace{\|\tilde{y}_h - y_h\|_{L^2(\Omega)}^2}_{K_2}.$$

K₁ :

Consider the following equation,

$$\begin{aligned} -\Delta t - \nabla \cdot (\vec{\beta}t) + ct &= y - \tilde{y}_h \text{ in } \Omega \\ t &= 0 \quad \text{on } \Gamma. \end{aligned}$$

By the boundedness of the bilinear form and using the Galerkin orthogonality,

$$\begin{aligned} \|y - \tilde{y}_h\|_{L^2(\Omega)}^2 &= a_h(y - \tilde{y}_h, t) \\ &= a_h(y - \tilde{y}_h, t - P_h t) + \underbrace{a_h(y - \tilde{y}_h, P_h t)}_0 \\ &\leq C \|t - P_h t\| \cdot \|y - \tilde{y}_h\| \leq Ch \|t\|_{H^2(\Omega)} \|y - \tilde{y}_h\|. \end{aligned}$$

By using Lemma 2.1.2 and Lemma 6.1.3, we obtain

$$K_1 \leq Ch \|y - \tilde{y}_h\|_{L^2(\Omega)} \|y\|_{H^1(\Omega)} \leq \frac{1}{4} \|y - \tilde{y}_h\|_{L^2(\Omega)}^2 + Ch^2 \|y\|_{H^1(\Omega)}^2.$$

By canceling $\|y - \tilde{y}_h\|_{L^2(\Omega)}^2$, we obtain that

$$K_1 \leq Ch^2 \|y\|_{H^1(\Omega)}^2.$$

K₂ :

Let us define another dual equation,

$$\begin{aligned} -\Delta v - \nabla \cdot (\vec{\beta}v) + cv &= \tilde{y}_h - y_h \text{ in } \Omega \\ v &= 0 \quad \text{on } \Gamma. \end{aligned}$$

$$\begin{aligned}
\|\tilde{y}_h - y_h\|_{L^2(\Omega)}^2 &= a_h(\tilde{y}_h - y_h, v) \\
&= \underbrace{a_h(\tilde{y}_h - y, v)}_{K_{21}} + \underbrace{a_h(y - y_h, v)}_{K_{22}}.
\end{aligned}$$

K₂₁ :

Likewise K_1 ,

$$\begin{aligned}
K_{21} &= a_h(\tilde{y}_h - y, v) = a_h(\tilde{y}_h - y, v - P_h v) + \underbrace{a_h(\tilde{y}_h - y, P_h v)}_0 \\
&\leq C\|v - P_h v\| \|\tilde{y}_h - y\| \leq Ch\|v\|_{H^2(\Omega)} \|\tilde{y}_h - y\|.
\end{aligned}$$

By using Lemma 2.1.2 and Lemma 6.1.3, we obtain

$$K_{21} \leq Ch\|\tilde{y}_h - y_h\|_{L^2(\Omega)} \|y\|_{H^1(\Omega)}.$$

K₂₂ :

$$K_{22} = a_h(y - y_h, v) = \underbrace{a_h(y - y_h, v - P_h v)}_{K_{221}} + \underbrace{a_h(y - y_h, P_h v)}_{K_{222}}.$$

By using the boundedness of the bilinear form, Lemma 2.1.2 and Lemma 6.1.3,

$$\begin{aligned}
K_{221} &= a_h(y - y_h, v - P_h v) \leq C\|v - P_h v\| \|y - y_h\| \\
&\leq Ch\|v\|_{H^2(\Omega)} \|y - y_h\| \leq Ch\|\tilde{y}_h - y_h\|_{L^2(\Omega)} \|y - y_h\|.
\end{aligned}$$

Thus,

$$K_{221} \leq Ch\|\tilde{y}_h - y_h\|_{L^2(\Omega)} \|y - y_h\|.$$

By using Lemma 6.1.7, we obtain

$$\begin{aligned}
K_{221} &\leq Ch \|\tilde{y}_h - y_h\|_{L^2(\Omega)} \|y - y_h\| \\
&\leq Ch \|\tilde{y}_h - y_h\|_{L^2(\Omega)} (h^{-1/2} \|q - q_h\|_{L^2(\Gamma)} + \|y\|_{H^1(\Omega)}) \\
&\leq C \|\tilde{y}_h - y_h\|_{L^2(\Omega)} (h^{1/2} \|q - q_h\|_{L^2(\Gamma)} + h \|y\|_{H^1(\Omega)}).
\end{aligned}$$

\mathbf{K}_{222} :

Using the fact that $v = 0$ on Γ and Lemma 6.1.3, we have

$$\begin{aligned}
K_{222} &= a_h(y - y_h, P_h v) = \ell_h(0; q - q_h, P_h v) \leq Ch^{-1} \|q - q_h\|_{L^2(\Gamma)} \|P_h v\|_{L^2(\Gamma)} \\
&\leq Ch^{-1} \|q - q_h\|_{L^2(\Gamma)} \|P_h v - v\|_{L^2(\Gamma)} \leq Ch^{-1} \|q - q_h\|_{L^2(\Gamma)} h^{3/2} \|v\|_{H^2(\Omega)},
\end{aligned}$$

where the trace inequality and Lemma 2.1.2 are used, we obtain

$$K_{222} \leq Ch^{1/2} \|q - q_h\|_{L^2(\Gamma)} \|\tilde{y}_h - y_h\|_{L^2(\Omega)}.$$

Thus, we have

$$\begin{aligned}
\|\tilde{y}_h - y_h\|_{L^2(\Omega)}^2 &\leq K_{21} + \underbrace{K_{22}}_{K_{221} + K_{222}} \\
&\leq Ch^2 \|y\|_{H^1(\Omega)} + C \|\tilde{y}_h - y_h\|_{L^2(\Omega)} (h^{1/2} \|q - q_h\|_{L^2(\Gamma)} + h \|y\|_{H^1(\Omega)}) \\
&\leq \frac{1}{4} \|\tilde{y}_h - y_h\|_{L^2(\Omega)}^2 + Ch^2 \|y\|_{H^1(\Omega)}^2 + Ch \|q - q_h\|_{L^2(\Gamma)}^2.
\end{aligned}$$

By canceling $\|\tilde{y}_h - y_h\|_{L^2(\Omega)}^2$, we obtain

$$\|\tilde{y}_h - y_h\|_{L^2(\Omega)}^2 \leq Ch^2 \|y\|_{H^1(\Omega)}^2 + Ch \|q - q_h\|_{L^2(\Gamma)}^2.$$

Finally, we obtain

$$\begin{aligned}\|y - y_h\|_{L^2(\Omega)}^2 &\leq \|y - \tilde{y}_h\|_{L^2(\Omega)}^2 + \|\tilde{y}_h - y_h\|_{L^2(\Omega)}^2 \\ &\leq C(h\|q - q_h\|_{L^2(\Gamma)}^2 + h^2\|y\|_{H^1(\Omega)}^2).\end{aligned}$$

By taking the square root, we conclude

$$\|y - y_h\|_{L^2(\Omega)} \leq C(h^{1/2}\|q - q_h\|_{L^2(\Gamma)} + h\|y\|_{H^1(\Omega)}).$$

□

6.1.1 Main Result

Now, we are ready to prove the main result of the thesis.

Theorem 6.1.8. Let Ω be a convex polygonal domain, \bar{q} be the optimal control of the problem (1.2.1) and \bar{q}_h be its optimal SIPG solution. Then, for h sufficiently small, there exists a positive constant C independent of h such that

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \leq Ch^{1/2}(\|\bar{q}\|_{H^{1/2}(\Gamma)} + \|\bar{y}\|_{H^1(\Omega)} + \|\hat{y}\|_{L^2(\Omega)}). \quad (6.1.3)$$

Proof. Since \bar{q} is the optimal solution of (1.2.1) and \bar{q} satisfies the equation (5.0.3b), we have

$$\alpha\langle \bar{q}, \phi_h \rangle_\Gamma + \langle \phi_h, \frac{\partial \bar{z}}{\partial n} \rangle_\Gamma = 0 \quad \forall \phi_h \in L^2(\Gamma). \quad (6.1.4)$$

Since \bar{q}_h is the approximate solution of (1.2.1) and \bar{q}_h satisfies the equation (5.0.4b), we

have

$$\alpha \langle \bar{q}_h, \phi_h \rangle_\Gamma + \langle \phi_h, \frac{\bar{z}_h}{\partial n} \rangle_\Gamma - \frac{\gamma}{h} \langle \phi_h, \bar{z}_h \rangle_\Gamma - \langle \bar{z}_h | \vec{n} \cdot \vec{\beta} |, \phi_h \rangle_{\Gamma^-} = 0 \quad \forall \phi_h \in Q_h. \quad (6.1.5)$$

Subtracting (6.1.4) from (6.1.5), for any $\phi_h \in Q_h$, we have

$$\alpha \langle \bar{q} - \bar{q}_h, \phi_h \rangle_\Gamma + \langle \phi_h, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \rangle_\Gamma + \frac{\gamma}{h} \langle \phi_h, \bar{z}_h \rangle_\Gamma + \langle \bar{z}_h | \vec{n} \cdot \vec{\beta} |, \phi_h \rangle_{\Gamma^-} = 0. \quad (6.1.6)$$

Taking $\phi_h = P_h^\partial(\bar{q} - \bar{q}_h) = P_h^\partial \bar{q} - P_h^\partial \bar{q}_h = P_h^\partial \bar{q} - \bar{q}_h$ in (6.1.6) and splitting

$$P_h^\partial \bar{q} - \bar{q}_h = (P_h^\partial \bar{q} - \bar{q}) + (\bar{q} - \bar{q}_h),$$

we obtain

$$\begin{aligned} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}^2 &= \alpha \langle \bar{q} - \bar{q}_h, \bar{q} - \bar{q}_h \rangle \\ &\leq \boxed{\alpha \langle \bar{q} - \bar{q}_h, P_h^\partial \bar{q} - \bar{q} \rangle_\Gamma}_{J_1} + \boxed{\langle P_h^\partial \bar{q} - \bar{q}, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \rangle_\Gamma}_{J_2} \\ &\quad + \boxed{\frac{\gamma}{h} \langle P_h^\partial \bar{q} - \bar{q}, \bar{z}_h \rangle}_{J_3} + \boxed{\langle P_h^\partial \bar{q} - \bar{q}, \bar{z}_h | \vec{n} \cdot \vec{\beta} | \rangle_{\Gamma^-}}_{J_4} \\ &\quad + \boxed{\langle \bar{q} - \bar{q}_h, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \rangle_\Gamma}_{J_5} + \boxed{\frac{\gamma}{h} \langle \bar{q} - \bar{q}_h, \bar{z}_h \rangle_\Gamma}_{J_6} \\ &\quad + \boxed{\langle \bar{q} - \bar{q}_h, \bar{z}_h | \vec{n} \cdot \vec{\beta} | \rangle_{\Gamma^-}}_{J_7} \\ &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \end{aligned} \quad (6.1.7)$$

Now, we shall estimate each term separately. Most terms can be estimated by using the

estimate of the L^2 -projection. However, the term $(\bar{z} - \bar{z}_h)$ in J_2 and J_5 is not in the discrete space, so additional arguments are needed to treat these terms.

Estimate for J_1

By the Cauchy-Schwarz inequality and using Lemma (6.1.4),

$$J_1 = \alpha \langle \bar{q} - \bar{q}_h, P_h^\partial \bar{q} - \bar{q} \rangle_\Gamma \leq \alpha \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \|P_h^\partial \bar{q} - \bar{q}\|_{L^2(\Gamma)} \leq C_1 h^{1/2} |\bar{q}|_{H^{1/2}(\Gamma)} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)},$$

where C_1 depends on α .

Estimates for J_3 and J_6

Using 6.1.5 to estimate $\|\bar{z}_h\|_{L^2(\Gamma)}$, the Cauchy-Schwarz inequality and the regularity of \bar{y} , then we have

$$\begin{aligned} J_3 &= \frac{\gamma}{h} \langle P_h^\partial \bar{q} - \bar{q}, \bar{z}_h \rangle_\Gamma \leq \frac{\gamma}{h} \|P_h^\partial \bar{q} - \bar{q}\|_{L^2(\Gamma)} \|\bar{z}_h\|_{L^2(\Gamma)} \\ &\leq C_3 h^{-1} h^{1/2} |\bar{q}|_{H^{1/2}(\Gamma)} h^{3/2} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\ &\leq C_3 h |\bar{q}|_{H^{1/2}} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\ &\leq C_3 h |\bar{q}|_{H^{1/2}} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}). \end{aligned}$$

Likewise,

$$\begin{aligned} J_6 &= \frac{\gamma}{h} \langle \bar{q} - \bar{q}_h, \bar{z}_h \rangle_\Gamma \leq \frac{\gamma}{h} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \|\bar{z}_h\|_{L^2(\Gamma)} \\ &\leq C_6 h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\ &\leq C_6 h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}), \end{aligned}$$

where C_3 and C_6 depend on γ .

Estimates for J_4 and J_7 :

By using the Cauchy-Schwarz inequality and the Lemma 6.1.5, we have

$$\begin{aligned}
J_4 &= \langle P_h^\partial \bar{q} - \bar{q}, \bar{z}_h | \vec{n} \cdot \vec{\beta} | \rangle_{\Gamma^-} \leq C_4 h^{1/2} |\bar{q}|_{H^{1/2}(\Gamma)} \|\beta\|_{L^\infty(\Gamma)}^{1/2} \|\bar{z}_h\|_{L^2(\Gamma)} \\
&\leq C_4 h^2 |q|_{H^{1/2}(\Gamma)} \|\beta\|_{L^\infty(\Gamma)}^{1/2} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\
&\leq C_4 h^2 |\bar{q}|_{H^{1/2}(\Gamma)} \|\beta\|_{L^\infty(\Gamma)}^{1/2} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}).
\end{aligned}$$

Likewise,

$$\begin{aligned}
J_7 &= \langle \bar{q} - \bar{q}_h, \bar{z}_h | \vec{n} \cdot \vec{\beta} | \rangle_{\Gamma^-} \leq C_7 \|\beta\|_{L^\infty(\Gamma)}^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \|\bar{z}_h\|_{L^2(\Gamma)} \\
&\leq C_7 \|\beta\|_{L^\infty(\Gamma)}^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} h^{3/2} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\
&\leq C_7 h^{3/2} \|\beta\|_{L^\infty(\Gamma)}^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}).
\end{aligned}$$

Estimate for J_5 :

By the Cauchy-Schwarz inequality, we have

$$J_5 = \langle \bar{q} - \bar{q}_h, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \rangle_\Gamma \leq \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \left\| \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \right\|_{L^2(\Gamma)}.$$

Let us define $\tilde{z}_h \in V_h$ to be the SIPG solution to \bar{z} i.e. $a_h(\chi, \tilde{z}_h) = (\hat{y} - \bar{y}, \chi) \forall \chi \in V_h$.

In particular, $a_h(\chi, \bar{z} - \tilde{z}_h) = 0$ by the Galerkin orthogonality. Thus, we continue as following,

$$\langle \bar{q} - \bar{q}_h, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \rangle_\Gamma \leq \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \left(\underbrace{\left\| \frac{\partial(\bar{z} - \tilde{z}_h)}{\partial n} \right\|_{L^2(\Gamma)}}_{J_{51}} + \underbrace{\left\| \frac{\partial(\tilde{z}_h - \bar{z}_h)}{\partial n} \right\|_{L^2(\Gamma)}}_{J_{52}} \right).$$

J_{51} :

By the triangle inequality, we have

$$\left\| \frac{\partial(\bar{z} - \tilde{z}_h)}{\partial n} \right\|_{L^2(\Gamma)} \leq \underbrace{\left\| \frac{\partial(\bar{z} - P_h \bar{z})}{\partial n} \right\|_{L^2(\Gamma)}}_{J_{511}} + \underbrace{\left\| \frac{\partial(P_h \bar{z} - \tilde{z}_h)}{\partial n} \right\|_{L^2(\Gamma)}}_{J_{512}}.$$

J_{511} :

By the trace inequality and Theorem 2.1.2, we obtain

$$\begin{aligned} J_{511} &= \left\| \frac{\partial(\bar{z} - P_h \bar{z})}{\partial n} \right\|_{L^2(\Gamma)}^2 = \sum_{e \in \Gamma} \left\| \frac{\partial(\bar{z} - P_h \bar{z})}{\partial n} \right\|_{L^2(e)}^2 \\ &\leq \sum_{\tau \in T_h} (Ch^{-1} \|\bar{z} - P_h \bar{z}\|_{H^1(\tau)}^2 + Ch \|\bar{z} - P_h \bar{z}\|_{H^2(\tau)}^2) \\ &\leq \sum_{\tau \in T_h} Ch \|\bar{z}\|_{H^2(\tau)}^2 = Ch \|\bar{z}\|_{H^2(\Omega)}^2 \leq Ch \|\hat{y} - \bar{y}\|_{L^2(\Omega)}^2. \end{aligned}$$

Thus,

$$J_{511} = \left\| \frac{\partial(\bar{z} - P_h \bar{z})}{\partial n} \right\|_{L^2(\Gamma)} \leq Ch^{1/2} \|\hat{y} - \bar{y}\|_{L^2(\Omega)}.$$

J_{512} :

Since $(P_h^\partial \bar{z} - \tilde{z}_h) \in V_h$, we can apply the trace theorem for discrete function and by using the inverse inequality, we obtain that

$$\begin{aligned} \left\| \frac{\partial(P_h^\partial \bar{z} - \tilde{z}_h)}{\partial n} \right\|_{L^2(\Gamma)} &\leq Ch^{-1/2} \|P_h \bar{z} - \tilde{z}_h\|_{H^1(\Omega)} \\ &\leq Ch^{-1/2} \|P_h \bar{z} - \tilde{z}_h\| \leq Ch^{-1/2} (\|P_h \bar{z} - \bar{z}\| + \|\bar{z} - \tilde{z}_h\|) \\ &\leq Ch^{-1/2} h \|\bar{z}\|_{H^2(\Omega)} \leq Ch^{1/2} \|\hat{y} - \bar{y}\|_{L^2(\Omega)}, \end{aligned}$$

where we have used Lemma 6.1.3 for $k = 1$ and Theorem 2.1.2.

Thus,

$$\left\| \frac{\partial(P_h \bar{z} - \tilde{z}_h)}{\partial n} \right\|_{L^2(\Gamma)} \leq Ch^{1/2} \|\hat{y} - \bar{y}\|_{L^2(\Omega)}.$$

Since $J_{51} = J_{511} + J_{512}$, we obtain

$$J_{51} = \left\| \frac{\partial(\bar{z} - \tilde{z}_h)}{\partial n} \right\|_{L^2(\Gamma)} \leq Ch^{1/2} \|\hat{y} - \bar{y}\|_{L^2(\Omega)} \leq Ch^{1/2} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)})_{L^2(\Omega)}.$$

J_{52} :

Since we have

$$a_h(\chi, \bar{z}_h) = (\hat{y} - \bar{y}_h, \chi)$$

$$a_h(\chi, \tilde{z}_h) = (\hat{y} - \bar{y}, \chi),$$

where $\forall \chi \in V_h$. We obtain

$$a_h(\chi, \tilde{z}_h - \bar{z}_h) = (\bar{y}_h - \bar{y}, \chi) \quad \forall \chi \in V_h. \quad (6.1.8)$$

Now, let us define the following equation

$$\begin{aligned} -\Delta w - \nabla \cdot (\vec{\beta} w) + cw &= \bar{y}_h - \bar{y} \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \Gamma. \end{aligned}$$

By using (6.1.8),

$$a_h(\chi, \tilde{z}_h - \bar{z}_h) = a_h(\chi, \tilde{z}_h) - a_h(\chi, \bar{z}_h) = (\hat{y} - \bar{y}, \chi) - (\hat{y} - \bar{y}_h, \chi) = (\bar{y}_h - \bar{y}, \chi) = a_h(\chi, w_h).$$

The above equality shows that $w_h = \tilde{z}_h - \bar{z}_h$.

Now, using the inverse inequality and the fact that $w = 0$ on Γ , we obtain

$$\begin{aligned}
\left\| \frac{\partial(\tilde{z}_h - \bar{z}_h)}{\partial n} \right\|_{L^2(\Gamma)} &\leq Ch^{-1} \|\tilde{z}_h - \bar{z}_h\|_{L^2(\Gamma)} = Ch^{-1} \|\tilde{z}_h - \bar{z}_h - w\|_{L^2(\Gamma)} \\
&\leq Ch^{-1} h^{1/2} \underbrace{\|\tilde{z}_h - \bar{z}_h - w\|}_{w_h} \leq Ch^{-1/2} \|w_h - w\| \leq Ch^{-1/2} h \|w\|_{H^2(\Omega)} \\
&\leq Ch^{1/2} \|y_h - \bar{y}\|_{L^2(\Omega)},
\end{aligned}$$

where we have used Theorem 2.1.2 and Lemma 6.1.3 for $k = 1$ in the last step.

Thus,

$$J_{52} = \left\| \frac{\partial(\tilde{z}_h - \bar{z}_h)}{\partial n} \right\|_{L^2(\Gamma)} \leq Ch^{1/2} \|\bar{y}_h - \bar{y}\|_{L^2(\Omega)}.$$

Finally, we obtain

$$J_5 \leq \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} (J_{51} + J_{52}) \leq C_5 h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)}).$$

Estimate for J_2 :

By using the Cauchy-Schwarz inequality, Lemma 6.1.4 and the estimate of $\left\| \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \right\|_{L^2(\Gamma)}$ in J_5 , we have

$$\begin{aligned}
J_2 &= \langle P_h^\partial \bar{q} - \bar{q}, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \rangle_\Gamma \leq \|P_h^\partial \bar{q} - \bar{q}\|_{L^2(\Gamma)} \left\| \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \right\|_{L^2(\Gamma)} \\
&\leq C_2 h^{1/2} \|\bar{q}\|_{H^{1/2}(\Gamma)} \left\| \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n} \right\|_{L^2(\Gamma)} \\
&\leq C_2 h^{1/2} \|\bar{q}\|_{H^{1/2}(\Gamma)} h^{1/2} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)}) \\
&= C_2 h \|\bar{q}\|_{H^{1/2}(\Gamma)} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)}).
\end{aligned}$$

Thus,

$$J_2 \leq Ch|q|_{H^{1/2}(\Gamma)}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)})$$

After using Lemma 6.1.7 to estimate $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$ and combining $J_1, J_2, J_3, J_4, J_5, J_6, J_7$ in (6.1.7), we obtain

$$\begin{aligned} \alpha\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\leq C_1 h^{1/2} |\bar{q}|_{H^{1/2}(\Gamma)} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \\ &+ C_2 h |\bar{q}|_{H^{1/2}(\Gamma)} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_3 h |\bar{q}|_{H^{1/2}(\Gamma)} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h \|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_4 h^2 |\bar{q}|_{H^{1/2}(\Gamma)} \|\beta\|_{L^\infty(\Gamma)}^{1/2} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h \|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_5 h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} (\|\hat{y}\|_{H^1(\Omega)} + \|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_6 h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} (\|\hat{y}\|_{H^1(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h \|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_7 h^{3/2} \|\beta\|_{L^\infty(\Gamma)}^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h \|\bar{y}\|_{H^1(\Omega)}). \end{aligned}$$

Notice that we can rewrite the above inequality as

$$\begin{aligned} \alpha\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}^2 &\leq C_I h |\bar{q}|_{H^{1/2}(\Gamma)}^2 + \frac{\alpha}{4} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}^2 \\ &+ C_{II} h (\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h \|\bar{y}\|_{H^1(\Omega)})^2 + \frac{\alpha}{4} \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}^2 \\ &+ C_{III} h \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}^2. \end{aligned}$$

After all simplification, we obtain

$$\alpha\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}^2 \leq Ch(|\bar{q}|_{H^{1/2}(\Gamma)} + \|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)})^2 + C'h\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}^2$$

where h is sufficiently small such that $C'h \leq \frac{\alpha}{2}$ to absorb $C'h\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}^2$ to the left hand

side. Thus, we conclude that there exists a positive constant C such that

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \leq Ch^{1/2}(|\bar{q}|_{H^{1/2}(\Gamma)} + \|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)})$$

provided h is sufficiently small.

□

Chapter 7

Numerical Examples

In this section, we show some numerical examples to support our theoretical results by the method described for the problem

$$\min_{\{y,q\}} J(y,q) = \frac{1}{2} \|y - \hat{y}\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|q\|_{L^2(\Gamma)}^2$$

subject to

$$\begin{aligned} -\epsilon \Delta y + \vec{\beta} \cdot \nabla y + cy &= f \quad \text{in } \Omega, \\ y &= q \quad \text{on } \Gamma, \end{aligned}$$

where Ω is a domain and Γ is its boundary. Here, we present numerical results depending on different kinds of domain as following.

1. Ω is a line segment

Since the domain is one dimensional and the boundary is consisting of two points, there is no regularity limitation due to geometry restriction. Thus, we do not expect

an optimal rate, but the method is stable and convergent.

By setting $\Omega = [0, 1]$, $\epsilon = 1$, $\vec{\beta} = [1]$, $\bar{q} = (1 - x)^2(x^2)$, $c = 0$, $\bar{y} = x^4 - \frac{e^{\frac{1-x}{\epsilon}} - e^{\frac{-1}{\epsilon}}}{1 - e^{\frac{-1}{\epsilon}}}$, and $\bar{z} = \frac{\alpha}{\epsilon}(1 - x)^2x^2$.

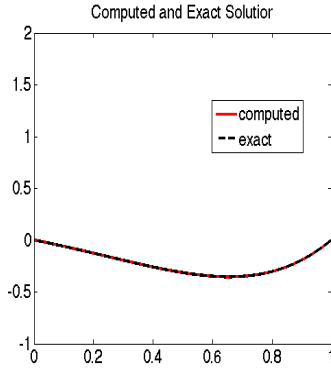


FIGURE 7.0.1: Computed and Exact Solution

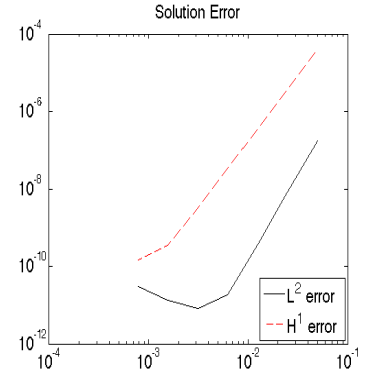


FIGURE 7.0.2: Solution Error

Error Rates for piecewise linear $ibasis = 1$.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	Left- q_{rate}	Right- q_{rate}
5.00e-01	1.959	1.002	2.007	1.965
2.50e-01	1.979	1.001	2.006	1.982
1.25e-01	1.990	1.001	2.004	1.991
6.25e-02	1.995	1.000	2.002	1.996
3.12e-02	1.997	1.001	2.001	1.998
1.56e-03	1.999	1.000	2.001	1.999

Error Rates for piecewise quadratic $ibasis = 2$.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	Left- q_{rate}	Right- q_{rate}
5.00e-01	2.999	2.013	2.025	2.982
2.50e-01	2.999	2.007	2.683	2.991
1.25e-01	3.000	2.004	2.864	2.996
6.25e-02	3.000	2.002	2.937	2.998
3.12e-02	2.998	2.001	2.969	2.999
1.56e-03	1.938	2.001	2.985	2.999

Error Rates for piecewise cubic $ibasis = 3$.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	Left- q_{rate}	Right- q_{rate}
5.00e-01	4.436	2.436	3.970	3.866
2.50e-01	4.425	3.423	3.985	3.983
1.25e-01	4.380	3.385	3.992	3.991
6.25e-02	1.188	3.320	3.996	3.996
3.12e-02	-0.777	3.223	3.998	3.998
1.56e-03	-1.119	1.268	3.999	3.999

2. Ω is a unit square domain

By setting

$$\begin{aligned}\vec{\beta} &= [1; 1] \text{ and } c = 1, \\ \bar{q} &= \frac{-1}{\epsilon}(x(1-x) + y(1-y)), \\ \bar{y} &= \frac{-1}{\epsilon}(x(1-x) + y(1-y)), \\ \bar{z} &= \frac{1}{\epsilon}(xy(1-x)(1-y)).\end{aligned}$$

- Numerical results for $\epsilon \gg h$ (the regular case) on the unit square domain.

Since the domain is square with the largest interior angle $p = \frac{\pi}{2}$, the expected convergence rate has been given in [29] as $\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \leq Ch$. By Theorem 6.1.7, $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq Ch^{3/2}$, and so $\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \leq Ch^{1/2}$. From Lemma 6.1.1 and 6.1.5, $\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \leq Ch^2$ and so $\|\bar{z} - \bar{z}_h\|_{H^1(\Omega)} \leq Ch$.

Error for ϵ on the unit square domain.

h	$\ y - y_{ex}\ _{L^2}$	$\ y - y_{ex}\ _{H^1}$	$\ q - q_{ex}\ _{L^2}$	$\ z - z_{ex}\ _{L^2}$	$\ z - z_{ex}\ _{H^1}$
5.00e-01	1.92e-01	3.91e+00	1.07e+00	4.31e-02	9.19e-01
2.50e-01	8.44e-02	2.25e+00	4.86e-01	1.38e-02	5.06e-01
1.25e-01	3.14e-02	1.27e+00	2.29e-01	4.21e-03	2.62e-01
6.25e-02	1.10e-02	7.33e-01	1.09e-01	1.22e-03	1.32e-01
3.12e-02	3.80e-03	4.49e-01	5.20e-02	3.32e-04	6.66e-02

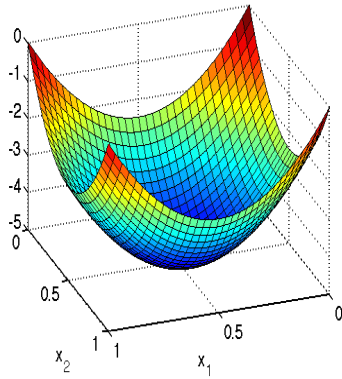


FIGURE 7.0.3: Exact State for $\epsilon \gg h$

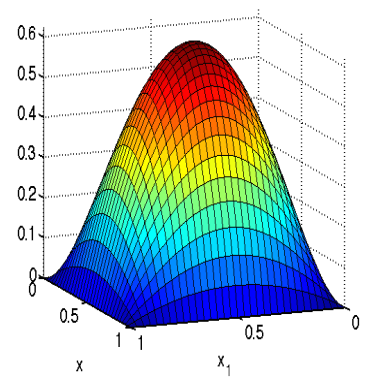


FIGURE 7.0.5: Exact Adjoint for $\epsilon \gg h$

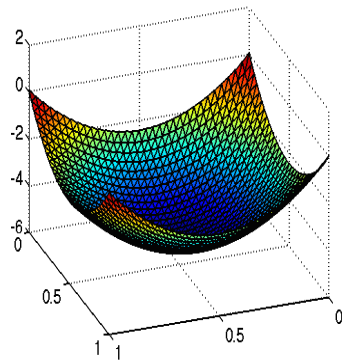


FIGURE 7.0.4: Computed State for $\epsilon \gg h$

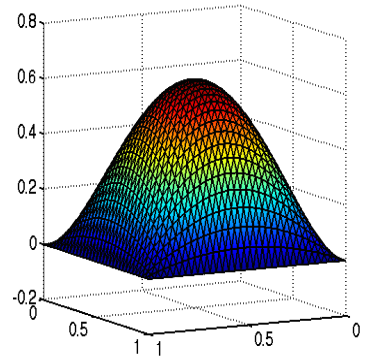


FIGURE 7.0.6: Computed Adjoint for $\epsilon \gg h$

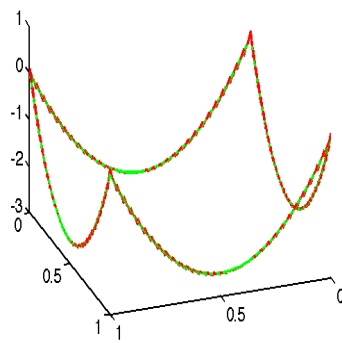


FIGURE 7.0.7: Exact and Computed Control for $\epsilon \gg h$

Error Rates for ϵ on the unit square domain.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	1.19	0.80	1.13	1.64	0.86
1.25e-01	1.43	0.83	1.08	1.72	0.95
6.25e-02	1.51	0.79	1.08	1.79	0.98
3.12e-02	1.53	0.71	1.06	1.87	0.99
expected	1.50	0.50	1.00	2.00	1.00

- Numerical results for $h \gg \epsilon$ (the advection-diffusion dominated case) on the unit square domain.

Since ϵ is too small for this case, we have the advection-diffusion dominated case and the norm of y depends on ϵ such that $\|\bar{y}\|_{H^{k+1}(\Omega)} \leq \frac{C}{\epsilon^{k+1/2}}$. Since the convergence rate of \bar{q} depends on data of \bar{y} from the main result, we do not expect any convergence rate. However, surprisingly we obtain some convergence rate. Also, it can be seen that some oscillatory solutions and nonconvergent rate of q appears on the inflow boundary whereas it is stable interior of the boundary because the Dirichlet boundary condition is almost ignored by the method as a result of weak treatment and it does not resolve the layers and causes oscillations on the boundary.

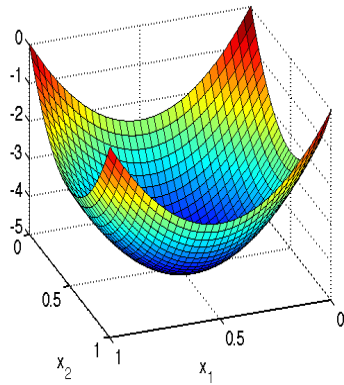


FIGURE 7.0.8: Exact State for $h \gg \epsilon$

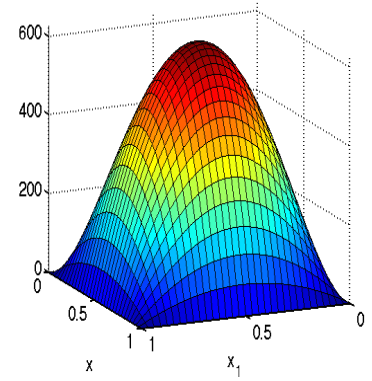


FIGURE 7.0.10: Exact Adjoint for $h \gg \epsilon$

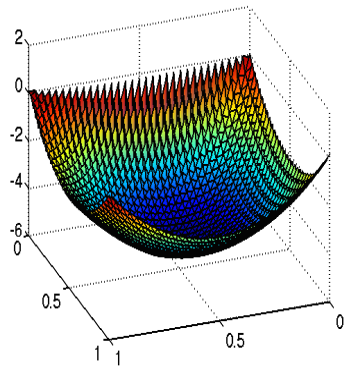


FIGURE 7.0.9: Computed State for $h \gg \epsilon$

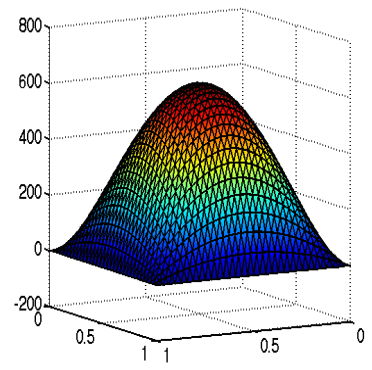


FIGURE 7.0.11: Computed Adjoint for $h \gg \epsilon$

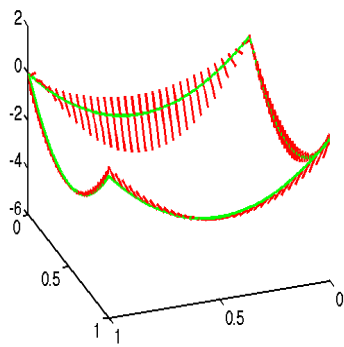


FIGURE 7.0.12: Exact and Computed Control for $h \gg \epsilon$

Error for $h \gg \epsilon$ on the unit square domain.

h	$\ y - y_{ex}\ _{L^2}$	$\ y - y_{ex}\ _{H^1}$	$\ q - q_{ex}\ _{L^2}$	$\ z - z_{ex}\ _{L^2}$	$\ z - z_{ex}\ _{H^1}$
5.00e-01	4.29e+00	2.68e+01	8.15e+00	5.66e+01	9.52e+02
2.50e-01	7.69e-01	1.32e+01	2.22e+00	1.51e+01	5.15e+02
1.25e-01	4.72e-01	1.58e+01	9.70e-01	3.85e+00	2.63e+02
6.25e-02	3.78e-01	2.36e+01	6.66e-01	9.64e-01	1.32e+02
3.12e-02	2.55e-01	2.88e+01	6.37e-01	2.40e-01	6.62e+01

Error Rates for $h \gg \epsilon$ on the unit square domain.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	2.48	1.02	1.87	1.91	0.89
1.25e-01	0.71	-0.26	1.20	1.97	0.97
6.25e-02	0.32	-0.58	0.54	2.00	0.99
3.12e-02	0.57	-0.28	0.06	2.01	1.00

3. Ω is a diamond shaped domain

By a transformation from the unit square domain to obtain a diamond shaped domain Ω with $\frac{\pi}{4}, \pi/8$ and $\pi/10$ angles, while the angle of the domain is getting smaller, we expect that the error rate is getting close to the predicted optimal error rate to confirm our main result.

- Numerical results for $h \ll \epsilon$ on the diamond shape domain with angle $\pi/4$.

Error Rates for $h \ll \epsilon$ on the diamond shape domain with angle $\pi/4$

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	1.51	0.80	0.65	1.43	1.78
1.25e-01	1.47	0.84	0.96	1.75	1.91
6.25e-02	1.69	0.85	1.05	1.89	1.96
3.12e-02	1.75	0.81	1.05	1.95	1.98
1.56e-02	1.73	0.74	1.03	1.97	1.99

- Numerical results for $h \gg \epsilon$ on the diamond shape domain with angle $\pi/4$.

While the method still works, currently we cannot explain the following rates and we will consider it as a future work.

Error Rates for $h \gg \epsilon$ on the diamond shape domain with angle $\pi/4$

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	3.35	2.45	3.36	2.87	1.89
1.25e-01	0.31	0.23	1.41	2.96	1.97
6.25e-02	-0.01	-0.64	0.81	2.97	1.99
3.12e-02	0.01	-0.68	0.45	2.45	2.00

- Numerical results for $h \ll \epsilon$ on the diamond shape domain with angles $\pi/8$ and $\pi/10$.

By a transformation from the unit square domain to obtain a diamond shaped domain Ω with $\frac{\pi}{8}$ and $\frac{\pi}{10}$ angles, while the regularity of state is reducing sharply

close to the predicted rate to confirm our main result, the regularity of control and adjoint which surprisingly is not reduced need another investigation.

Error Rates for $h \ll \epsilon$ on the diamond shape domain with angle $\pi/8$

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	1.75	0.72	0.32	1.11	0.76
1.25e-01	1.16	0.41	0.72	1.63	0.95
6.25e-02	1.60	0.50	0.97	1.88	0.99
3.12e-02	1.70	0.55	1.01	1.96	1.00
1.56e-02	1.66	0.53	1.02	1.99	1.00

Error Rates for $h \ll \epsilon$ on the diamond shape domain with angle $\pi/10$

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	0.85	0.17	0.17	1.03	0.61
1.25e-01	0.92	0.17	1.59	1.60	1.04
6.25e-02	1.32	0.26	0.90	1.89	1.14
3.12e-02	1.50	0.42	1.00	1.98	1.10
1.56e-02	1.55	0.50	1.02	2.00	1.05

7.0.2 Conclusion and Future Work

In this thesis, we consider the Dirichlet boundary optimal control problem governed by the advection-diffusion equation and apply the DG methods to the problem. We show some attractive features of the method such as the stable behavior of SIPG method into the domain of the smoothness and for the advection dominated case except on the boundary as a result of the boundary weak treatments. We have proven that the convergence rate for the SIPG is optimal in the interior of the general convex domain. For general polygonal domains and Laplace equations it has been shown [29] that

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \leq Ch^{1-\frac{1}{p}},$$

where $p > 2$ depends on the largest angle. Also, it has been shown for nonsmooth data in [2],[22], [13], and the nonconvex domain in [3]. The optimal error rate for the problem with control and state constraints or the problem with subject to another types of PDEs are less predictable so needs to be future research.

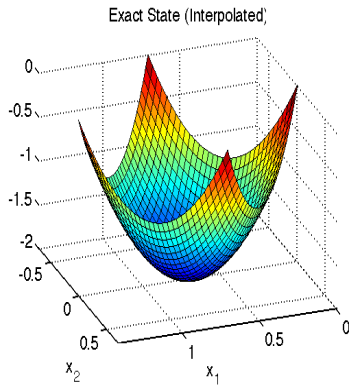


FIGURE 7.0.13: Exact State for $h \ll \epsilon$ on the diamond with angle $\pi/4$

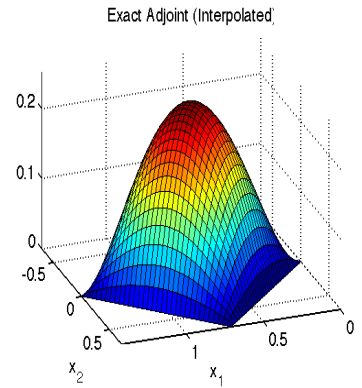


FIGURE 7.0.15: Exact Adjoint for $h \ll \epsilon$ on the diamond with angle $\pi/4$

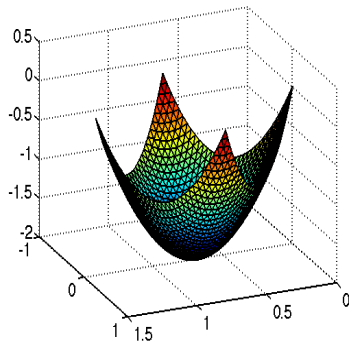


FIGURE 7.0.14: Computed State for $h \ll \epsilon$ on the diamond with angle $\pi/4$

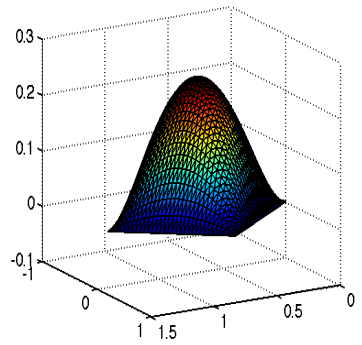


FIGURE 7.0.16: Computed Adjoint for $h \ll \epsilon$ on the diamond with angle $\pi/4$

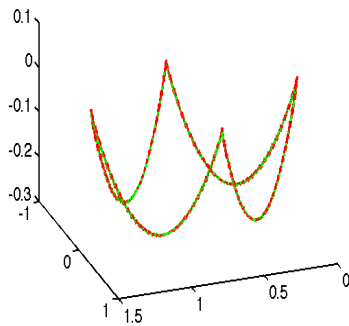


FIGURE 7.0.17: Exact and Computed Control for $h \ll \epsilon$ on the diamond with angle $\pi/4$

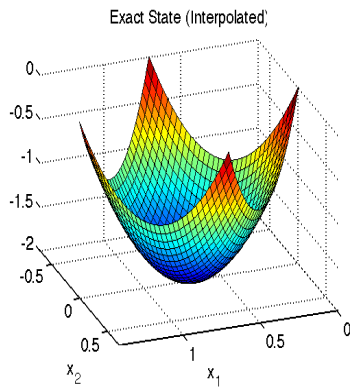


FIGURE 7.0.18: Exact State for $h \gg \epsilon$ on the diamond with angle $\pi/4$

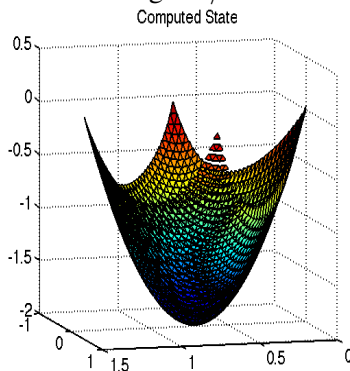


FIGURE 7.0.19: Computed State for $h \gg \epsilon$ on the diamond with angle $\pi/4$

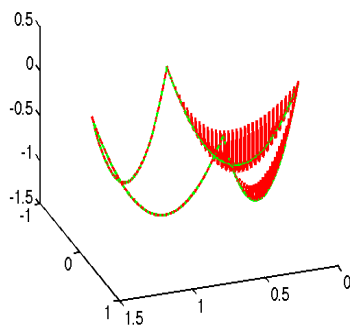


FIGURE 7.0.22: Exact and Computed Control for $h \gg \epsilon$ on the diamond with angle $\pi/4$

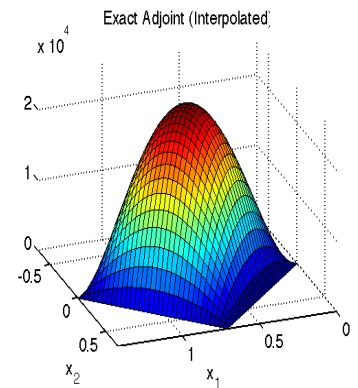


FIGURE 7.0.20: Exact Adjoint for $h \gg \epsilon$ on the diamond with angle $\pi/4$

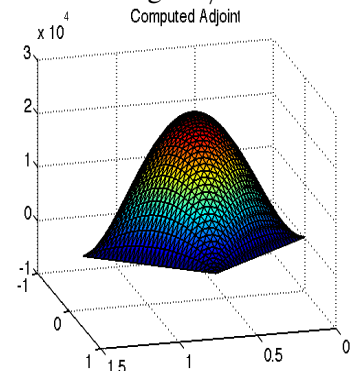


FIGURE 7.0.21: Computed Adjoint for $h \gg \epsilon$ on the diamond with angle $\pi/4$

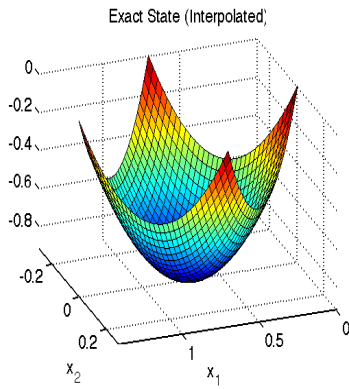


FIGURE 7.0.23: Exact State
for $h \gg \epsilon$ on the diamond with
angle $\pi/8$

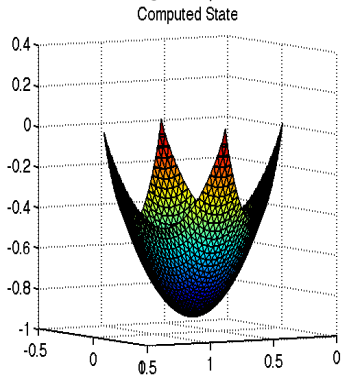


FIGURE 7.0.24: Computed
State for $h \gg \epsilon$ on the diamond
with angle $\pi/8$

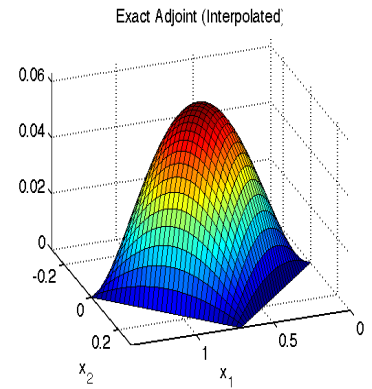


FIGURE 7.0.25: Exact Adjoint
for $h \gg \epsilon$ on the diamond with
angle $\pi/8$

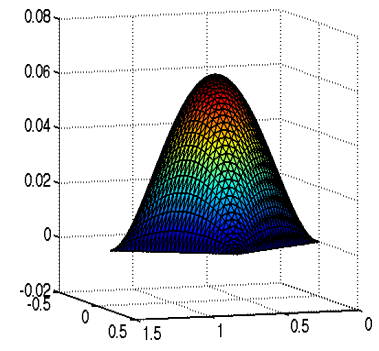


FIGURE 7.0.26: Computed
Adjoint for $h \gg \epsilon$ on the
diamond with angle $\pi/8$

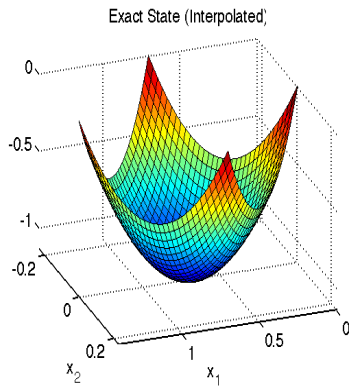


FIGURE 7.0.27: Exact State for $h \gg \epsilon$ on the diamond with angle $\pi/10$

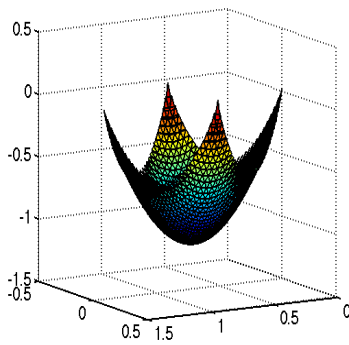


FIGURE 7.0.28: Computed State for $h \gg \epsilon$ on the diamond with angle $\pi/10$

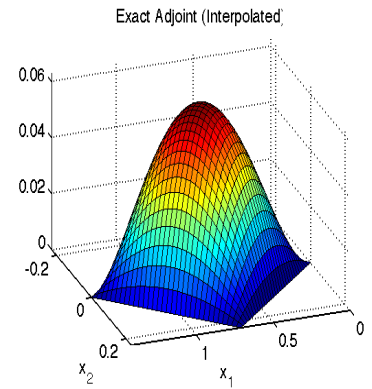


FIGURE 7.0.29: Exact Adjoint for $h \gg \epsilon$ on the diamond with angle $\pi/10$

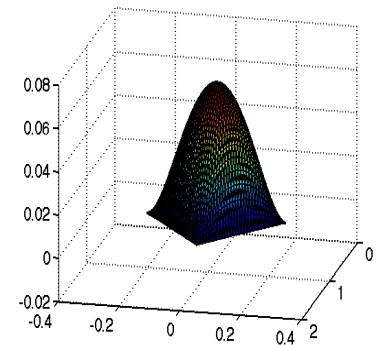


FIGURE 7.0.30: Computed Adjoint for $h \gg \epsilon$ on the diamond with angle $\pi/10$

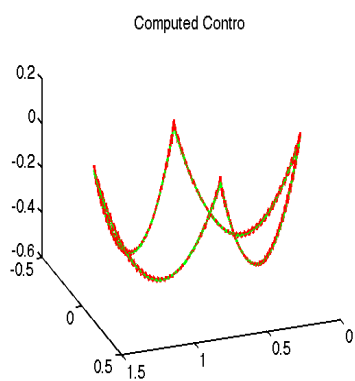


FIGURE 7.0.31: Exact and Computed Control for $h \ll \epsilon$ on the diamond with angle $\pi/8$

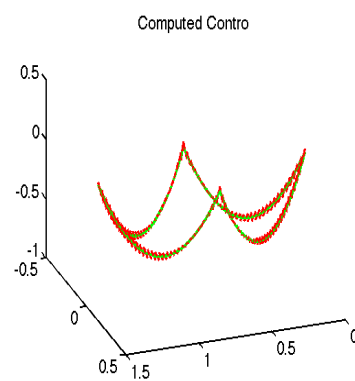


FIGURE 7.0.32: Exact and Computed Control for $h \ll \epsilon$ on the diamond with angle $\pi/10$

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